We present a compact atomic clock interrogating ultracold $^{87}$Rb magnetically trapped on an atom chip. Very long coherence times sustained by spin self-rephasing allow us to interrogate the atomic transition with 85% contrast at 5-s Ramsey time. The clock exhibits a fractional frequency stability of $5.8 \times 10^{-13}$ at 1 s and is likely to integrate into the $10^{-15}$ range in less than a day. A detailed analysis of seven noise sources explains the measured frequency stability. Fluctuations in the atom temperature (0.4 nK shot-to-shot) and in the offset magnetic field ($5 \times 10^{-6}$ relative fluctuations shot-to-shot) are the main noise sources together with the local oscillator, which is degraded by the 30% duty cycle. The analysis suggests technical improvements to be implemented in a future second generation setup. The results demonstrate the remarkable degree of technical control that can be reached in an atom chip experiment.

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I. INTRODUCTION

Atomic clocks are behind many everyday tasks and numerous fundamental science tests. Their performance has made a big leap through the discovery of laser cooling [1–3] giving one the ability to control the atom position on the millimeter scale. It has led to the development of atomic fountain clocks [4,5] which have reached a stability limited only by fundamental physics properties, i.e., quantum projection noise and Fourier-limited linewidth [6]. While these laboratory-size setups are today’s primary standards, mobile applications such as telecommunication, satellite-aided navigation [7] or spacecraft navigation [8] call for smaller instruments with liter-scale volume. In this context, it is natural to consider trapped atoms. The trap overcomes gravity and thermal expansion and thereby enables further gain on the interrogation time.

Typical storage times of neutral atoms range from a few seconds to minutes [9,10]. Thus a trapped-atom clock with long interrogation times could measure energy differences in the mHz range in one single shot. Hence, if trap-induced fluctuations can be kept low, trapped atoms could not only define time with this resolution, but could also be adapted to measure other physical quantities like electromagnetic fields, accelerations or rotations with very high sensitivity.

A founding step towards very long interrogation of trapped neutral atoms was made in our group through the discovery of spin self-rephasing [11] which sustains several tens of seconds coherence time [11–13]. This rivals trapped-ion clocks, the best of which has shown 65-s interrogation time and a stability of $2 \times 10^{-14}$ at 1 s [14,15].

It is to be compared to compact clocks using thermal vapor and buffer gas [16–18] or laser cooled atoms [19–22]. Among these the record stability is $1.6 \times 10^{-13}$ at 1 s [17]. Clocks with neutral atoms trapped in an optical lattice have reached impressive stabilities down to the $10^{-18}$ range [23,24] but their interrogation time is so far limited by the local oscillator. Research into making such clocks transportable is ongoing [25,26]. We describe the realization of a compact clock using neutral atoms trapped on an atom chip and analyze trap-induced fluctuations.

Our “trapped-atom clock on a chip” (TACC) employs laser cooling and evaporative cooling to reach ultracold temperatures where neutral atoms can be held in a magnetic trap. Realizing a 5-s Ramsey time, we obtain 100-mHz linewidth and 85% contrast on the hyperfine transition of $^{87}$Rb. We measure the fractional frequency stability as $5.8 \times 10^{-13}\tau^{-1/2}$. It is reproduced by analyzing several noise contributions, in particular atom number, temperature, and magnetic field fluctuations. The compact setup is realized through the atom chip technology [27], which builds on the vast knowledge of microfabrication. The use of atom chips is also widespread for the study of Bose-Einstein condensates [9,28], degenerate Fermi gases [29], and gases in low dimensions [30,31]. Other experiments strive for the realization of quantum information processors [32–34]. The high sensitivity and micron-scale position control have been used for probing static magnetic [35] and electric [36] fields as well as microwaves [37]. Creating atom interferometers [38] on atom chips is equally appealing. Here, an on-chip high stability atomic clock not only provides an excellent candidate for mobile timing applications, it also takes a pioneering role among this broad range of atom chip experiments, demonstrating that experimental parameters can be mastered to the fundamental physics limit.

This paper is organized as follows: We first describe the atomic levels and the experimental setup. Then we give the evaluation of the clock stability and an analysis of all major noise sources.

II. ATOMIC LEVELS

We interrogate the hyperfine transition of $^{87}$Rb (Fig. 1). A two-photon drive couples the magnetically trappable states $|1\rangle \equiv |F = 1, m_F = -1\rangle$ and $|2\rangle \equiv |F = 2, m_F = 1\rangle$, whose transition frequency exhibits a minimum at a magnetic field near $B_m \approx 3.229$ G [39,40]. This second-order dependence strongly reduces the clock frequency sensitivity to...
magnetic field fluctuations. It assures that atoms with different trajectories within the trap still experience similar Zeeman shifts. Furthermore, by tuning the offset magnetic field, the inhomogeneity from the negative collisional shift [40] can be compensated to give a quasi-position-invariant overall shift [41]. Under these conditions of strongly reduced inhomogeneity we have shown that spin self-rephasing can overcome dephasing and that coherence times of $58 \pm 12\ s$ [11] can be reached. It confirms the possibility to create a high stability clock [42].

### III. EXPERIMENTAL SETUP

The experimental setup, details of which are given in [43], is similar to compact atom chip experiments reported previously [44,45]. All experimental steps, laser cooling, evaporative cooling, interrogation, and detection take place in an $\sim (5\ cm)^3$ glass cell where one cell wall is replaced by the atom chip (Fig. 2). In this first-generation setup, a 25-l/s ion pump is connected via standard vacuum components. It evacuates the cell to a pressure of $\sim 1 \times 10^{-9}\ \text{mbar}$. The cell is surrounded by a 10 $\times\ 10\ \times\ 15\ cm^3$ cage of Helmholtz coils. A 30-cm diameter optical table holds the coil cage as well as all beam expanders necessary for cooling and detection and is surrounded by two layers of magnetic shielding.

The timing sequence (Table I) starts with a mirror MOT [44] loading $\sim 3 \times 10^6$ atoms in $\sim 7\ s$ from the background vapor. The MOT magnetic field is generated by one of the coils and a U-shaped copper structure placed behind the atom chip [46]. Compressing the MOT followed by 5 ms optical molasses cools the atoms to $\sim 20\ \mu\text{K}$. The cloud is then optically pumped to the $|1\rangle$ state and transferred to the magnetic trap. It is gradually compressed to perform RF evaporation, which takes $\sim 3\ s$. A 0.7-s decompression ramp transfers the atoms to the final interrogation trap with trap frequencies $\omega_x,\omega_y,\omega_z > (2.7, 9, 74)\ Hz$ located $350 cm$ below the surface. It is formed by two currents on the chip and two currents in two pairs of Helmholtz coils. The currents are supplied by home-built current supplies with relative stability $< 10^{-5}$ at 3 A [47]. The final atom number is $N = 2 - 4 \times 10^4$ and their temperature $T \sim 80\ n\text{K}$. The density is thus with $\bar{n} \approx 1.5 \times 10^{11}\ \text{atoms/cm}^3$ so low that the onset of Bose-Einstein condensation would occur at 5 nK. With $k_BT/\hbar\omega_x,\omega_y,\omega_z > 20$ the ensemble can be treated by the Maxwell-Boltzmann distribution.

![FIG. 2. (Color online) (a) The implemented atom chip. One identifies the Z-shaped coplanar waveguide which serves for atom trapping and transport of the microwave interrogation signal. The outer dimensions are $38 \times 45.5\ mm^2$. (b) The chip constitutes one of the facets of the vacuum cell facilitating electrical contact. The cell is surrounded by a $10 \times 10 \times 15\ cm^3$ cage of Helmholtz coils and a 30-cm diameter optical table holding all optical beam expanders. The cell is evacuated via standard UHV equipment.](image)

![FIG. 1. (Color online) Level scheme of the $^{87}\text{Rb}$ ground state. The two magnetically trappable clock states $|F = 1,m_F = -1\rangle$ and $|F = 2,m_F = 1\rangle$ are coupled via a two-photon, microwave and radiofrequency transition, where the microwave is tuned $500\ kHz$ above the $|1, -1\rangle \rightarrow |2, 0\rangle$ transition.](image)

### Table I. Timing sequence of one experimental cycle. The total cycle time is 16 s.

<table>
<thead>
<tr>
<th>Operation</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOT</td>
<td>6.85 s</td>
</tr>
<tr>
<td>Compressed MOT</td>
<td>20 ms</td>
</tr>
<tr>
<td>Optical molasses</td>
<td>5 ms</td>
</tr>
<tr>
<td>Optical pumping</td>
<td>1 ms</td>
</tr>
<tr>
<td>Magnetic trapping and compression</td>
<td>230 ms</td>
</tr>
<tr>
<td>RF evaporation</td>
<td>3 s</td>
</tr>
<tr>
<td>Magnetic decompression</td>
<td>700 ms</td>
</tr>
<tr>
<td>First Ramsey pulse</td>
<td>77.65 ms</td>
</tr>
<tr>
<td>Ramsey time</td>
<td>5 s</td>
</tr>
<tr>
<td>Second Ramsey pulse</td>
<td>77.65 ms</td>
</tr>
<tr>
<td>Time of flight $</td>
<td>1\rangle,</td>
</tr>
<tr>
<td>Detection</td>
<td>20 $\mu$s</td>
</tr>
</tbody>
</table>
phase noise is detailed in Sec. V B. The RF signal of variable frequency \( f_{\text{RF}} \sim 2 \text{ MHz} \) comes from a commercial DDS which supplies a “standard” wire parallel to the waveguide. The two-photon Rabi frequency is about \( \Omega = 3.2 \text{ Hz} \) making a \( \pi/2 \) pulse last \( \tau_p \) = 77.65 ms \( \gg 2\pi/\Omega \). The pulse duration is chosen so that any Rabi frequency inhomogeneity, which was characterized in [50], is averaged out and Rabi oscillations show 99.5% contrast. Two pulses enclose a Ramsey time of \( T_R = 5 \text{ s} \). Detection is performed via absorption imaging. A strongly saturating beam crosses the atom cloud and is imaged onto a back illuminated, high quantum efficiency CCD camera with frame transfer (Andor iKon M 934-BRDD). Illumination (20 \( \mu \text{s} \)) without and with repump light, 5.5 ms and 8.5 ms after trap release, probes the \( F = 2 \) and \( F = 1 \) atoms independently. Between these two, a transverse laser beam blows away the \( F = 2 \) atoms. Numerical frame re-composition generates the respective reference images and largely reduces the effect of optical fringes [51]. Calculation of the optical density and correction for the high saturation [52] give access to the atom column density. The so found two-dimensional (2D) atom distributions are fitted by Gaussians to extract the number of atoms in each state \( N_{1,2} \). The transition probability is calculated as \( P = N_2/(N_1 + N_2) \) accounting for total atom number fluctuations. The actual detection noise is discussed in Sec. V A. The total time of one experimental cycle is \( T_c = 16 \text{ s} \).

IV. STABILITY MEASUREMENT

Prior to any stability measurement we record the typical Ramsey fringes. We repeat the experimental cycle while scanning \( \nu_{\text{LO}} = \nu_{\text{MW}} + \nu_{\text{RF}} \) over \( \sim 3 \) fringes. Doing so for various Ramsey times \( T_R \) allows one to identify the central fringe corresponding to the atomic frequency \( \nu_{\text{at}} \). Figure 3 shows typical fringes for \( T_R = 5 \text{ s} \), where each point is a single shot. One recognizes the Fourier limited linewidth of 100 mHz equivalent to \( \sim 10^{11} \) quality factor. The 85% contrast is remarkable. A sinusoidal fit gives the slope at the fringe half-height \( dP/d\nu = 13.4/\text{Hz} \), which is used in the following stability evaluation to convert the detected transition probability into frequency.

Evaluation of the clock stability implies repeating the experimental cycle several thousand times. The clock is free running, i.e., we measure the transition probability at each cycle, but we do not feedback to the interrogation frequency \( \nu_{\text{LO}} \). Only an alternation in successive shots from a small fixed negative to positive detuning, \( \Delta_{\text{mad}}/(2\pi) = \pm 50 \text{ mHz} \), probes the left and right half-height of the central fringe. The difference in \( P \) between two shots gives the variation of the central frequency independent from long-term detection or microwave power drifts. In the longest run, we have repeated the frequency measurement over 18 h.

The measured frequency data is traced in Fig. 4 versus time. Besides shot-to-shot fluctuations one identifies significant long-term variations. Correction of the data with the atom number, by a procedure we will detail in Sec. V C 1, results in substantial improvement. We analyze the data by the Allan standard deviation which is defined as [53]

\[
\sigma_y^2(\tau) \equiv \frac{1}{2} \sum_{k=1}^{[L/2^l] - 1} (\tilde{y}_{k+1} - \tilde{y}_k)^2.
\]

Here \( L \) is the total number of data points and the \( \tilde{y}_k \) are averages over packets of \( 2^l \) successive data points with \( l \in [0, 1, \ldots, [\log_2 L]] \) and \( \tau = 2^l T_c \). Figure 5 shows the Allan standard deviation of the uncorrected and corrected data. For \( 0 \leq l \leq 9 \) the points and their error bars are plotted as calculated with the software \textsc{Stable32} [54]. This software uses Eq. (1) to find the points. The error bars are calculated as the 5%–95% confidence interval based on the appropriate \( \chi^2 \) distribution. The software stops output at \( l = [\log_2 L] - 2 \) since there are too few differences \( \tilde{y}_{l+1} - \tilde{y}_l \) to give a statistical error bar. Instead we directly plot all differences for \( l = 10 \) and 11.

The Allan standard deviation shows the significant improvement brought by the atom number correction. The uncorrected data starts at \( \tau = T_c = 16 \text{ s} \) with \( \sigma_z = 1.9 \times 10^{-13} \).
TABLE II. List of identified contributions to the clock (in)stability. Atom temperature fluctuations dominate followed by magnetic field fluctuations and local oscillator noise. The quadratic sum of all contributions explains the measured stability.

<table>
<thead>
<tr>
<th>Relative frequency stability (10^{-15})</th>
<th>Shot-to-shot</th>
<th>At 1 s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured, without correction</td>
<td>2.0</td>
<td>7.2</td>
</tr>
<tr>
<td>Measured, after N correction</td>
<td>1.5</td>
<td>5.8</td>
</tr>
<tr>
<td>Atom temperature</td>
<td>1.0</td>
<td>3.9</td>
</tr>
<tr>
<td>Magnetic field</td>
<td>0.7</td>
<td>2.6</td>
</tr>
<tr>
<td>Local oscillator</td>
<td>0.7</td>
<td>2.7</td>
</tr>
<tr>
<td>Quantum projection</td>
<td>0.4</td>
<td>1.5</td>
</tr>
<tr>
<td>N correction</td>
<td>0.4</td>
<td>1.5</td>
</tr>
<tr>
<td>Atom loss</td>
<td>0.3</td>
<td>1.1</td>
</tr>
<tr>
<td>Detection</td>
<td>0.3</td>
<td>1.1</td>
</tr>
<tr>
<td>Total estimate</td>
<td>1.5</td>
<td>6.0</td>
</tr>
</tbody>
</table>
We quantify the above three noise types from an independent measurement: Only the first $\pi/2$ pulse is applied and $P$ is immediately detected. The measurement is repeated for various atom numbers and $\sigma_p(N)$ is extracted. Figure 6 shows the measured $\sigma_p$ shot-to-shot versus $N$. Considering the noise sources as statistically independent, we fit the data by $\sigma_p^2 = \sigma_{\text{det}}^2/(2N^3) + 1/(4N) + \sigma_{P,\text{RF+IR}}^2$ and find $\sigma_{\text{det}} = 59$ atoms and $\sigma_{P,\text{RF+IR}} < 10^{-4}$. The fit gives $\sigma_{\text{det}} = 59$ atoms and $\sigma_{P,\text{RF+IR}} < 10^{-4}$.

Because of the periodicity of $\psi_t$, $\sigma_{\text{det}}(v_{\text{LO}}N)^{-1} |dP/dv_t|^2 = \sigma_{\text{det}}(v_{\text{LO}}N)^{-1} |dP/dv_t|^2 = 0.3 \times 10^{-13}$ shot-to-shot. In both we have used $dP/dv_t$ as measured in Fig. 3.

### B. Local oscillator noise

The experimental cycle probes $v_{\text{at}} - v_{\text{LO}}$ only during the Ramsey time. Atom preparation and detection cause dead time. Repeating the experimental cycle then constitutes periodic sampling of the LO frequency and its fluctuations. This, as is well known from numerical data acquisition, leads to aliasing. It folds high Fourier frequency LO noise close to multiples of the sampling frequency $1/T_c$ back to low frequency variations, which degrade the clock stability. Thus even high Fourier frequency noise can degrade the clock signal. The degradation is all the more important as the dead time is long and the duty cycle $d = T_R/T_c$ is low. This stability degradation $\sigma_{\text{Dick}}$ is known as the Dick effect [58]. It is best calculated using the sensitivity function $g(t)$ [59]: During dead time, $g = 0$ whereas during $T_R$, when the atomic coherence $|\psi_t| = (1 + e^{i\theta}/2)|/\sqrt{2}$ is fully established $g = 1$. During the first Ramsey pulse, when the coherence builds up, $g$ increases as $\sin \Omega t$ for a square pulse and decreases symmetrically for the second pulse [60]. Then the interrogation outcome is

$$\delta v = \frac{\int_{-T_c/2}^{T_c/2} (v_{\text{at}}(t) - v_{\text{LO}}(t)) g(t) dt}{\int_{-T_c/2}^{T_c/2} g(t) dt}, \quad (2)$$

with

$$g(t) = \begin{cases} a \sin \Omega t & 0 \leq t < T_c/2 \\ a \sin \Omega (T_c/2 + T_p - t) & T_c/2 \leq t \leq T_c + \pi - \tau_p \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Typically $\Omega \tau_p = \pi/2$ and, for operation at the fringe half-height, $a = \sin \Delta T_c R_0 = 1$. Because of the periodicity of repeated clock measurements, it is convenient to work in Fourier space with

$$g(t) = \frac{1}{T_c} \int_{-T_c/2}^{T_c/2} g(t) \cos(2\pi f t/T_c) dt. \quad (4)$$

Using the power spectral density of the LO frequency noise $S_f^R(f)$ (see Fig. 7), the contribution to the clock stability becomes the quadratic sum over all harmonics [59],

$$\sigma_{\text{Dick}}^2(\tau) = \frac{1}{\tau} \sum_{l=1}^{\infty} \left( \frac{g_l}{g_0} \right)^2 S_f^R(\tau/T_c). \quad (5)$$

Here we have assumed $v_{\text{at}}$ constant in time; its fluctuations are treated in the next section. The coefficients $(g_l/g_0)^2$ are shown as points in Fig. 8 for our conditions. The weight of the first few harmonics is clearly the strongest, rapidly decaying.
over six decades in the range $1/T_c \approx 0.1$ Hz to $1/\tau_p \approx 10$ Hz. Above $\sim 10$ Hz the $g_i$ are negligible.

To measure $S^i_y(f)$ we divide our LO into two principal components: the 100-MHz reference signal derived from the hydrogen maser and the frequency multiplication chain generating the 6.8-GHz interrogation signal. We characterize each independently by measuring the phase noise spectrum $S^i_\phi(f)$. The fractional frequency noise $S^i_y(f)$ is obtained from simple differentiation as $S^i_y(f) = f^2 S^i_\phi(f)/\nu^2_{\text{MW}}$ [59]. The frequency noise of the RF signal can be neglected as its relative contribution is 3 orders of magnitude smaller.

We characterize the frequency multiplication chain by comparing it to a second similar model also constructed in-house. The two chains are locked to a common 100-MHz reference and their phase difference at 6.834 GHz is measured in-house. The two chains are locked to a common 100-MHz reference signal derived from a cryogenic sapphire oscillator (CSO) [61,62]. Now the mixer is M/A-COM PD-121. The CSO is itself locked to the reference signal but with a time constant of $\sim 1000$ s [63]. This being much longer than our cycle time, we can, for our purposes, consider the two as free running. The CSO is known from prior analysis [64] to be at least 10 dB lower in phase noise than the reference signal for Fourier frequencies higher than 0.1 Hz. Thus the measured noise can be attributed to the reference signal for the region of the spectrum $f > 1/T_c$ where our clock is sensitive. The phase noise spectrum is shown in Fig. 7. For comparison it was scaled to 6.8 GHz by adding 37 dB. Several maxima characteristic of the several phase locks in the systems can be identified. At low Fourier frequencies, the reference signal noise is clearly above the chain noise. For all frequencies, both are well above the noise floor of our measurement system. The noise of the reference signal being dominant in the range $1/T_c$ to $1/\tau_p$, where our clock is sensitive, we neglect the chain noise in the following.

Using Eq. (5), we estimate the Dick effect contribution as $\sigma_{\text{Dick}} = 2.7 \times 10^{-13} \pi^{-1/2}$. This represents the second biggest contribution to the noise budget (Table II). It is due to the important dead time and the long cycle time which folds in the LO noise spectrum where it is strongest. Improvement is possible, first of all, through reduction of the dead time which is currently dominated by the $\sim 7$-s MOT loading phase and the 3-s evaporative cooling. Options for faster loading include pre-cooling in a 2D MOT [65] or a single-cell fast pressure modulation [66]. Utilization of a better local oscillator like the cryogenic sapphire oscillator seems obvious but defies the compact design. Alternatively, generation of low phase noise microwaves from an ultrastable laser and femtosecond comb has been demonstrated by several groups [67–69] and ongoing projects aim at miniaturization of such systems [70]. If a quartz local oscillator remains the preferred choice, possibly motivated by cost, one long Ramsey time must be divided into several short interrogation intervals interlaced by nondestructive detection [71–73].

C. Fluctuations of the atomic frequency

1. Atom number fluctuation

Having characterized the fluctuations of the LO frequency, we now turn to fluctuations of the atomic frequency. We begin by atom number fluctuations. Due to the trap confinement and the ultracold temperature, the atom density is 4 orders of magnitude higher than what is typically found in a fountain clock. Thus the effect of atom-atom interactions on the atomic frequency must be taken into account even though $^{87}\text{Rb}$ presents a substantially lower collisional shift than the standard $^{133}\text{Cs}$. Indeed, when plotting the measured clock frequency against the detected atom number $N = N_1 + N_2$, which fluctuates by 2%–3% shot-to-shot, we find a strong correlation (Fig. 9). The distribution is compatible with a linear fit with slope $k = -2.70(7) \mu\text{Hz/atom}$. In order to confirm this value with a theoretical estimate we use the mean-field approach and the s-wave scattering lengths $a_{ij}$ which depend on the atomic states only [40]:

$$\Delta v_C(\vec{r}) = \frac{2\hbar}{m} n(\vec{r}) [(a_{22} - a_{11}) + (2a_{12} - a_{11} - a_{22})\theta],$$

where $n(\vec{r})$ is the position-dependent density and $a_{11} = 100.44a_0$, $a_{22} = 95.47a_0$, and $a_{12} = 98.99a_0$ are the scattering lengths with $a_0 = 0.529 \times 10^{-10}$ m [40]. We assume perfect $\pi/2$ pulses and so $\theta \equiv (N_1 - N_2)/N = 0$. Integrating over the Maxwell-Boltzmann density distribution we get

$$\Delta v_C = N - \frac{\hbar(a_{11} - a_{22})\sqrt{\pi} a_0\omega_x\omega_z}{4(\pi k_B T)^{3/2}}.$$
We must consider that the atom number decays during the time of flight where the camera region cannot be corrected. The final atom number $N_f$ at the end of the Ramsey time is known from the detection, but the initial atom number $N_i$ can only be retraced with a statistical error. To estimate this contribution we first consider the decay from the initial atom number $N_i$. At time $t$, the probability for a given atom to still be trapped is $e^{-\gamma t}$ and the probability to have left the trap is $1 - e^{-\gamma t}$. Given $N_i$, the probability $p$ to have $N_f$ atoms at $t$ is proportional to $e^{-N_f/\gamma}(1 - e^{-\gamma t})^{N_i - N_f}$ to the number of possible combinations:

$$p(N_f \text{ given } N_i) = \frac{N_f!}{N_i!(N_i - N_f)!} e^{-N_f/\gamma}(1 - e^{-\gamma t})^{N_i - N_f}.$$  

The sum of this binomial distribution over all $0 \leq N_f \leq N_i$ is by definition normalized. We are interested in the opposite case: Since we detect $N_f$ at $t = T_R$, we search the probability of $N_i$ given $N_f$.

$$p(N_i \text{ given } N_f) = \frac{A N_f!}{N_f!(N_f - N_i)!} \times e^{-N_f/\gamma(T_R - t)}(1 - e^{-\gamma T_R})^{N_i - N_f}.$$  

The combinatorics are as in Eq. (9) when replacing $N_i \rightarrow N_f$ and $N_f \rightarrow N_i$, but now normalization sums over $0 \leq N_i < \infty$. Here it is convenient to approximate the binomial distribution by the normal distribution.

$$p(N_i \text{ given } N_f) \approx \frac{A}{\sqrt{2\pi \eta}} e^{-(N_f - N_i)e^{-\gamma(T_R - t)}(T_f - 1)/2}.$$  

with $\eta = N_f e^{-\gamma(T_R - t)}(1 - e^{-\gamma T_R})$ and hence $A = e^{-\gamma(T_R - t)}$. Then, the mean of $N_i$ is

$$\langle N_i \rangle = (N_f + 1)e^{\gamma T_R} - 1 \approx N_f e^{\gamma T_R},$$

and its statistical error,

$$\sigma_{N_i} = \sqrt{(1 - e^{\gamma(T_R - t)})(2 - (N_f + 2)e^{\gamma T_R})} \approx \sqrt{N_f(e^{\gamma T_R} - 1)e^{\gamma T_R}}.$$  

Setting $t = 0$, we get $\sigma_{N_i} = 210$. Integrating $\sigma_{N_i}$ over $T_R$ gives $\sigma_{N_i} = 113 \approx N_i/2$ and a frequency fluctuation of $\sigma_{\nu_{\text{los}}} = 0.3 \times 10^{-13}$ shot-to-shot. This can be improved by increasing the trap lifetime well beyond the Ramsey time, which for our setup implies better vacuum with lower background pressure. Alternatively one can perform a nondestructive measurement of the initial atom number [74]. Assuming an error of 80 atoms on such a detection would decrease the frequency noise to $\sigma_{\nu_{\text{los}}} = 0.1 \times 10^{-13}$ shot-to-shot.

2. Magnetic field and atom temperature fluctuations

We have analyzed the effect of atom number fluctuations. Two other parameters strongly affect the atomic frequency: the atom temperature and the magnetic field. We show that their influence can be evaluated by measuring the clock stability for different magnetic fields at the trap center. We begin by modeling the dependence of the clock frequency.

Our clock operates near the magic field $B_m \approx 3.229$ G for which the transition frequency has a minimum of $-4497.31$ Hz.
with respect to the field free transition,
\[ \Delta v_B(\vec{r}) = b(B(\vec{r}) - B_m)^2, \]
with \( b \approx 431 \text{ Hz/G}^2 \). For atoms trapped in a harmonic potential in the presence of gravity, the Zeeman shift becomes position dependent,
\[ \Delta v_B(\vec{r}) = \frac{b m^2}{\mu_B^2} \left( \omega_c^2 \chi^2 + \omega_c^2 \gamma^2 + \omega_c^2 z^2 - 2g z + \delta B \frac{\mu_B}{m} \right)^2, \]
with \( \delta B \equiv B(\vec{r}) - B_m \) and \( g \) the gravitational acceleration [41]. Using the Maxwell-Boltzmann distribution the ensemble averaged Zeeman shift is
\[ \Delta v_B = \frac{b}{\mu_B} \left( 4g^2 m k_B T + 15k_B T^2 \right) \]
\[ + 6 \mu_B \delta B k_B T + \delta B^2 \frac{\mu_B}{m}. \]
Differentiation with respect to \( \delta B \) leads to the effective magnetic field,
\[ \delta B_0^B = -3k_B T / \mu_B, \]
where the ensemble averaged frequency is independent from magnetic field fluctuations. For \( T = 80 \text{ nK} \), \( \delta B_0^B = -3.6 \text{ mG} \) whose absolute value almost coincides with the magnetic field inhomogeneity across the cloud, \( (B(\vec{r})^2 - B_m^2)^{1/2} = \sqrt{6k_B T} / \mu_B = 2.92 \text{ mG} \). \( \delta B_0^B \) is close to the field of maximum contrast \( \delta B_0^m \approx -40 \text{ mG} \) such that the fringe contrast is still 85% (Fig. 10).

If \( \delta B \neq \delta B_0^B \) is chosen the clock frequency fluctuations due to magnetic field fluctuations are
\[ \sigma_{y,B} = \frac{2b}{v_{\text{at}}} |\delta B_0^B - \delta B| \sigma_B. \]
We will use this dependence to measure \( \sigma_B \).

Temperature fluctuations affect the range of magnetic fields probed by the atoms and the atom density, i.e., the collisional shift. Differentiation of both with respect to temperature also leads to an extremum, where the clock frequency is insensitive to temperature fluctuations. The extremum puts a concurrent condition on the magnetic field with
\[ \delta B_0^T = -\frac{15k_B T + 2g^2 m}{3\mu_B} - \frac{\hbar(a_{11} - a_{22})(e^{\gamma T} - 1)^3 N_f \mu_B \mu_{\omega_2}}{16\pi^3 b(k_B T)^{3/2} T_R}. \]
For our conditions, \( \delta B_0^B = -3.6 \text{ mG} \) and \( \delta B_0^T = -79 \text{ mG} \) are not identical but close and centered around \( \delta B_0^C \). We will see in the following that a compromise can be found where the combined effect of magnetic field and temperature fluctuations is minimized. A “doubly magic” field cannot be found as always \( \delta B_0^T < \delta B_0^B \), but lower \( T \) reduces their difference. If \( \delta B \neq \delta B_0^C \) is chosen, the clock frequency fluctuations due to temperature fluctuations are
\[ \sigma_{y,T} = \frac{6b k_B}{\mu_B v_{\text{at}}} |\delta B_0^T - \delta B| \sigma_T, \]
thus varying \( \delta B \) allows to measure \( \sigma_T \), too.

We determine \( \sigma_B \) and \( \sigma_T \) experimentally by repeating several stability measurements for different \( \delta B \) over a range of 200 mG where the contrast is above 70%. The shot-to-shot stability is shown in Fig. 11. One identifies a clear minimum of the instability at \( \delta B \approx -35 \text{ mG} \), which coincides with \( \delta B_0^C \) and is a compromise between the two optimal points \( \delta B_0^T \) and \( \delta B_0^B \). This means that both magnetic field and temperature fluctuations are present with roughly equal weight. We model

FIG. 10. (Color online) Fringe contrast (top) and differential Zeeman shift (bottom) of the clock frequency with respect to the frequency minimum for various magnetic fields. \( \delta B = 0 \) indicates the magic field of 3.229 G. The contrast maximum is offset by \(-40 \text{ mG} \).
the data with a quadratic sum of all so far discussed noise sources. Most of them give a constant offset; the slight variation due to the contrast variation shown in Fig. 10 is negligible. σ_{\delta,T} and σ_{\delta,B} are fitted by adjusting σ_B and σ_T. We find shot-to-shot temperature fluctuations of σ_T = 0.44 nK or 0.55% relative to 80 nK. The shot-to-shot magnetic field fluctuations are σ_B = 16 µG or 5 × 10^{-6} in relative units. The values demonstrate our exceptional control of the experimental apparatus. Because the ambient magnetic field varies by <10 mG and the lowest magnetic shielding factor is 3950, we attribute σ_B to the instability of our current supplies. Indeed, it is compatible with the measured relative current stability [47].

The atom temperature fluctuations are small compared to a typical experiment using evaporative cooling. This may again be due to the exceptional magnetic field stability, since the atom temperature is determined by the magnetic field at the trap bottom during evaporation and the subsequent opening of the magnetic trap. At all stages, the current control is the most crucial. Using Eqs. (18) and (20), the temperature and magnetic field fluctuations translate into a frequency noise of σ_{\delta,T} = 1.0 × 10^{-13} and σ_{\delta,B} = 0.7 × 10^{-13} shot-to-shot, respectively. The comparison in Table II shows that these are the main sources of frequency instability together with the Dick effect. Therefore, improving the magnetic field and temperature noise is of paramount importance. The atom temperature can in principle be extracted from the absorption images, which we take at each shot. Analyzing the data set of Fig. 4 gives shot-to-shot fluctuations of σ_T / T = 2 – 4%, which is much bigger than the 0.55% deduced above. We therefore conclude that the determination of the cloud width is overshadowed by a significant statistical error. Nevertheless, it needs to be investigated, whether better detection and/or imaging at long time of flight, may reduce this error. The magnetic field stability may be improved by refined power supplies, the use of multiwire traps [75], microwave dressing [57] or ultimately the use of atom chips with permanent magnetic material [76–78]. If the magnetic field fluctuations can be reduced, the temperature fluctuations may also reduce. Small σ_B would also allow one to operate nearer δB_T^\textsuperscript{opt}.

VI. CONCLUSION

We have built and characterized a compact atomic clock using magnetically trapped atoms on an atom chip. The clock stability reaches 5.8 × 10^{-13} at 1 s and is likely to integrate into the 10^{-15} range in less than a day. This is similar to the performance of the best compact atomic microwave clocks under development. It furthermore demonstrates the high degree of technical control that can be reached with atom chip experiments. After correction for atom number fluctuations, variations of the atom temperature and magnetic field are the dominant causes of the clock instability together with the local oscillator noise. The magnetic field stability may be improved by additional current sensing and feedback and ultimately by the use of permanent magnetic materials. This would allow one to operate nearer the second sweet spot where the clock frequency is independent from temperature fluctuations. The local oscillator noise takes an important role, because the clock duty cycle is <30%. We are now in the process of designing a second version of this clock, incorporating fast atom loading and nondestructive atom detection. We thereby expect to reduce several noise contributions to below 1 × 10^{-13}×^{1/2}.

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