EXPANSION OF THE HAMILTONIAN OF A PLANETARY SYSTEM INTO THE POISSON SERIES IN ALL ORBITAL ELEMENTS

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ABSTRACT. The study of planetary systems orbital evolution is one of important problems of celestial mechanics. This work is the first stage in our investigation of this problem. We present algorithm for constructing of a planetary system Hamiltonian expansion into the Poisson series in all orbital elements. The expansion was constructed for a planetary systems with 4 planets. So, we can study orbital evolution of giant-planets of the Solar System and many extrasolar systems also. Estimation accuracy of Hamiltonian expansion is presented in this work.

1. INTRODUCTION

Let us consider planetary system with 4 planets. We need to write its Hamiltonian. For our purpose we can use canonical Jacobi coordinates (Murray, Dermott, 2009). It is hierarchical coordinate system, which is more preferable for investigation of planetary system orbital evolution. A position of each following body is determined relative to a center of inertia of previously including bodies set. We need to know differences of radius vectors in inertial frame. This frame can be barycentric for example. Differences are determined here:

$$|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j| = \mathbf{r}_i - \mathbf{r}_j + \mu \sum_{k=j}^{i-1} \frac{m_k}{\bar{m}_k} \mathbf{r}_k, \tag{1}$$

where numbers i and j satisfy a condition $1 \le j < i \le N$; $\boldsymbol{\rho}_k$ is barycentric radius vector of k-th planet, $\mathbf{r}_k$ is Jacobi radius vector of the same planet; μm_k is mass of the planet in items of Sun mass, $\bar{m}_k = 1 + \mu m_1 + \ldots + \mu m_k$, μ is small parameter. Variable μ denotes ratio of sum of planets masses to mass of the Sun. For the Solar system the value of μ can take equal to 0.001.

The Hamiltonian h can be expressed as sum of two terms – undisturbed part and disturbing function (Kholshevnikov et al., 2001), as shown here:

$$h = -\sum_{i=1}^{N} \frac{M_i \kappa_i^2}{2a_i} + \mu \times \frac{G m_0}{a_0} \Big\{ \sum_{i=2}^{N} \frac{a_0 m_i (2\mathbf{r}_i \mathbf{R}_i + \mu R_i^2)}{r_i \tilde{R}_i (r_i + \tilde{R}_i)} - \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{a_0 m_i m_j}{|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|} \Big\}, \tag{2}$$

where G is gravitational constant, a_0 is any constant of length typical for a planetary system (for example 1 astronomical unit), m_0 is mass of the Sun, M_i is normalized mass, κ_i^2 is gravitational parameter, a_i is semi-major axis; N is number of planets; other quantities are defined below:

$$\mathbf{R}_i = \sum_{k=1}^{i} \frac{m_k}{\bar{m}_k} \mathbf{r}_k, \quad \tilde{R}_i = \sqrt{r_i^2 + 2\mu \mathbf{r}_i \mathbf{R}_i + \mu^2 R_i^2}. \tag{3}$$

The first sum in (2) is undisturbed part of the Hamiltonian. The expression in figure brackets is the disturbing function. Introducing the value of a_0 into account, the disturbing function becomes dimensionless. Double sum in (2) is major part of the disturbing function. The major part describes interaction between planets. Denominator of the major part is defined in expression (1).

We used the second system of Poincare elements for constructing of the Hamiltonian expansion. It allows sufficiently simplifying an angular part of the series expansion. In this case only one angular element – mean longitude is defined.

After that, we get the Hamiltonian of a planetary system in this form:

$$h = h_0 + \sum_{k,n} A_{kn} x^k \cos(ny), \tag{4}$$

where h_0 is undisturbed Hamiltonian, A_{kn} is numerical coefficient, x^k is product of Poincare elements with corresponding degrees, cosine is represent an angular part of the series, ny is linear combination of mean longitudes of planets.

2. AGORITHM

Computer algebra system Piranha is used for expansion of the Hamiltonian. This program was written by Francesco Biscani (Biscani, 2009). Piranha is new specialized system for analytical calculations in celestial mechanics. It is multi-platform C++ program with Python's interface. At this moment Piranha is one of the fastest computer algebra systems. Piranha have various convenient implements for working with series. It allows set limit degree of series truncation, filtering of series items, substitution into series, saving to text files and others. Piranha works with different series types. In particular, supported series types are polynomials with rational numerical coefficients and Poisson series with polynomial coefficients.

Lets consider algorithm of constructing of the Hamiltonian expansion into the Poisson series:

- to be necessary make classical celestial mechanics series, such as x/a, y/a, y/a and r/a, a/r, which are base elements for the Hamiltonian expansion. We need to transform expressions for these expansions from Kepler elements (eccentricity and mean anomaly) to Poincare elements. We can use standart algorithms for it (Sharlier, 1966). Classical expansions in Kepler elements can be obtained using the Kepler processor implemented in Piranha;

- next, using x/a, y/a, z/a series it is possible to take the expansion of scalar product. Series for r_i/r_j ratio is obtained from expansions of r_i/a_i and a_j/r_j;

- inverse absolute value of radius vectors difference in Jacobi frame, which is denoted below as $1/\Delta_{ij}$, can be expanded into a series as follows. Write the definition of $1/\Delta_{ij}$:

$$1/\Delta_{ij} = |\mathbf{r}_i - \mathbf{r}_j|^{-1} = \frac{1}{r_j}\Big(1 + \Big(\frac{r_i}{r_j}\Big)^2 - 2\Big(\frac{r_i}{r_j}\Big)\cos H\Big)^{-\frac{1}{2}} = \frac{1}{r_j}\sum_{n=0}^{\infty}\Big(\frac{r_i}{r_j}\Big)^n P_n(\cos H), \qquad (5)$$

  where P_n is Legendre polynomial of n-th degree, H is angle between vectors $\mathbf{r}_i$ and $\mathbf{r}_j$. In (5) you can see the generating function of Legendre polynomials. So, we can expand $1/\Delta_{ij}$ into Poisson series, using series for $1/r_j$ and r_i/r_j. The series in Legendre polynomials absolutely converges when $|r_i/r_j| < 1$. In our case Legendre polynomials have not inner structure and saved in series as symbol variables. It allows reducing of number of expansion terms, necessary working memory and disk space;

- after that, we can take expansion of the Hamiltonian. Common form of items of the major part expansion up to the second degree of small parameter is shown here:

$$\frac{1}{|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|} = \frac{1}{\Delta_{ij}}\Big(1 - \frac{2\mu A_{ij} + \mu^2 B_{ij}}{\Delta_{ij}^2}\Big)^{-\frac{1}{2}} = \frac{1}{\Delta_{ij}} - \mu\frac{A_{ij}}{\Delta_{ij}^3} + \mu^2\Big(-\frac{B_{ij}}{\Delta_{ij}^3} + \frac{3}{4}\frac{A_{ij}^2}{\Delta_{ij}^5}\Big) + \ldots, \qquad (6)$$

  and here for items of the second part of the disturbing function:

$$\begin{aligned} \frac{2\mathbf{r}_i\mathbf{R}_i + \mu\mathbf{R}_i^2}{r_i\tilde{R}_i(r_i + \tilde{R}_i)} &= \frac{C_i + \mu D_i}{r_i^3\sqrt{1 + \frac{\mu C_i + \mu^2 D_i}{r_i^2}}\Big(1 + \sqrt{1 + \frac{\mu C_i + \mu^2 D_i}{r_i^2}}\Big)} = \\ &= \frac{C_i}{r_i^3} + \mu\Big(-\frac{3}{2}\frac{C_i^2}{r_i^5} + \frac{1}{2}\frac{D_i}{r_i^3}\Big) + \mu^2\Big(\frac{5}{2}\frac{C_i^3}{r_i^7} - \frac{3}{2}\frac{C_i D_i}{r_i^5}\Big) + \ldots, \end{aligned} \qquad (7)$$

$$A_{ij} = (\mathbf{r}_i - \mathbf{r}_j)\sum_{k=j}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k,\ B_{ij} = \Big(\sum_{k=j}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k\Big)^2,\ C_i = \mathbf{r}_i\sum_{k=1}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k,\ D_i = B_{i1}, \qquad (8)$$

  $1 \le j < i \le N$ in (6) and $2 \le i \le N$ in (7). In our case $N = 4$. So, using series for inverse distances, scalar products and quantities of $1/\Delta_{ij}$ with various degrees, we can construct items of the Hamiltonian expansion. Such quantities as small parameter μ and masses ratio $m_k/\bar{m}_k$ are used as symbol variables in series constructing.

3. RESULTS

Calculations were performed on Quad-core PC with 2600 MHz Core i5 processor and 8 Gb available memory. Computer algebra system Piranha using on Unix-like OS Ubuntu 14. Algorithms for series calculations was written as Python-modules of Piranha.

In the process, Piranha showed a high speed of calculations. Table 1 presents a time of series calculation, a number of its items and estimation accuracy for base series. Parameter n in the first column is a limit of degrees of eccentric and oblique Poincare elements. Results in the last column are correspond to series for $1/\Delta_{ij}$ with maximum degree of Legendre polynomials is equal to 35.

n	series	x/a	y/a	z/a	r/a	a/r	r_i/r_j	$\mathbf{r}_i{\cdot}\mathbf{r}_j$	$1/\Delta_{ij}$
6	time	0.5^s	0.5^s	0.5^s	0.5^s	0.5^s	0.5^s	1^s	40^s
	items	146	146	216	66	61	847	6282	32628
	accuracy	10^{-8}	10^{-8}	10^{-8}	10^{-9}	10^{-9}	10^{-8}	10^{-7}	$10^{-12} - 10^{-8}$
11	time	12^s	12^s	12^s	19^s	19^s	1^s	56^s	$12^m 44^s$
	items	792	792	2128	303	298	13548	228629	515291
	accuracy	10^{-14}	10^{-14}	10^{-14}	10^{-15}	10^{-15}	10^{-13}	10^{-12}	$10^{-13} - 10^{-9}$

Table 1: Calculation time, number of terms and estimation accuracy for base series.

The value of n is determined by required accuracy of expansion of the disturbing function. Rows which are named 'accuracy' consist relative differences between series expansion and accurate formula. In this work estimation accuracy of base series is determined for the Solar System giant-planets. Indexed values were calculated for all planetary pairs of Solar System. A wide range of values in some cells is obtained various estimations accuracy for planets pairs. The value of $1/\Delta_{ij}$ for the planetary pair "Uranus–Neptune" has the lowest accuracy. The best accuracy gives the planetary pair "Jupiter–Neptune".

The Hamiltonian expansion was constructed to 1 degree of small parameter. Maximum considered degree of eccentric and oblique Poincare elements is 6. Legendre polynomials are considered up to 35 degree.

Precision of the Hamiltonian approximation was calculated for the Solar system and 47 UMa, HD 69830 extrasolar systems also. Kepler elements for the Solar System are taken w.r.t. epoch J2000.0 and correspond to mean ecliptic. Orbital elements, such as semi-major axes, eccentricities and perigee arguments, and planets masses of above extrasolar systems are taken from http://www.exoplanet.eu. Planetary system of star HD 69830 is interesting in that it is compact with orbits eccentricities of the order of 0.1. Estimation accuracy of the series approximation is presented in the Table 2 for all items of the disturbing function. Columns which are named 'accuracy' consist relative differences (absolute values) between series expansion and accurate formula.

	Solar System		47 UMa star system		HD 69830 star system	
	series expansion	accuracy	series expansion	accuracy	series expansion	accuracy
i,j	the major part		the major part		the major part	
1,2	$6.247 \cdot 10^{-2}$	$2 \cdot 10^{-5}$	0.26590	$4 \cdot 10^{-5}$	$1.271 \cdot 10^{-2}$	$1 \cdot 10^{-5}$
1,3	$2.12 \cdot 10^{-3}$	$1 \cdot 10^{-5}$	0.31009	$5 \cdot 10^{-5}$	$5.9438 \cdot 10^{-3}$	$5 \cdot 10^{-7}$
1,4	$1.599 \cdot 10^{-3}$	$2 \cdot 10^{-6}$	–	–	–	–
2,3	$5.72 \cdot 10^{-4}$	$7 \cdot 10^{-6}$	0.08499	$2 \cdot 10^{-5}$	$4.10297 \cdot 10^{-3}$	$4 \cdot 10^{-8}$
2,4	$4.43 \cdot 10^{-4}$	$1 \cdot 10^{-6}$	–	–	–	–
3,4	$1.95 \cdot 10^{-4}$	$1 \cdot 10^{-6}$	–	–	–	–
i	the second part		the second part		the second part	
2	$1.58379 \cdot 10^{-2}$	$4 \cdot 10^{-7}$	0.04968	$7 \cdot 10^{-5}$	$3.01 \cdot 10^{-3}$	$2 \cdot 10^{-5}$
3	$9.5 \cdot 10^{-5}$	$5 \cdot 10^{-6}$	0.02549	$1 \cdot 10^{-5}$	$1.6471 \cdot 10^{-3}$	$7 \cdot 10^{-7}$
4	$7 \cdot 10^{-6}$	$5 \cdot 10^{-6}$	–	–	–	–
	whole disturbing function		whole disturbing function		whole disturbing function	
Σ	$8.526 \cdot 10^{-2}$	$2 \cdot 10^{-5}$	0.63676	$5 \cdot 10^{-5}$	$2.741 \cdot 10^{-2}$	$3 \cdot 10^{-5}$

Table 2: Precision of estimation of the disturbing function.

Table 2 shows that estimation accuracy of the disturbing function expansion is about 10^{-5} for the Solar System and different extrasolar systems. According to expression (2) the disturbing function must be multiplied by small parameter μ. After that, we can get estimation accuracy of the Hamiltonian expansion into series. It is about 10^{-8}.

4. CONCLUSION

We described algorithm for constructing of the Hamiltonian expansion of a planetary system with 4 planets into the Poisson series in all elements. The expansion was made to 6 degree of orbital elements and to 1 degree of small parameter. Estimation accuracy of the disturbing function is presented in this paper. Relative difference between series estimation and accurate formula is about 10^{-5} for the Solar System and extrasolar systems. So, the Poisson series for the Hamiltonian was constructed with precision about 10^{-8}. Now we are constructing the expansion for the Hamiltonian to 11 degree of orbital elements and 2 degree of small parameter.

5. REFERENCES

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