ON THE MINIMIZATION PROPERTIES OF TISSERAND SYSTEMS

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ABSTRACT. Tisserand systems are a useful concept to model the rotation of deformable sets of particles. They can be characterized by means of three alternative conditions related with the angular momentum and kinetic energy of the set. In this note, we revisit the issue providing a new proof of the equivalence between some of these defining conditions. In addition, we determine the time evolution of Tisserand systems in a clear way.

1. TISSERAND SYSTEMS. EQUIVALENT CONDITIONS

When a discrete or continuous collection of material particles $\mathcal{S}$ experience relative displacements, it is no possible to define unambiguously a rotational motion of the system – or *non-rigid body*. The usual solution is to assign to $\mathcal{S}$ a certain reference system $Oxyz$ with origin in the body barycenter O and linked to it in some prescribed way (the "body axes"). By doing so, the rotation of the system of particles is identified with the rotation of the body axes with respect to some inertial, or quasi–inertial, reference system $OXYZ$ (the "fixed axes"). This rotation admits a precise mathematical definition.

There are different possibilities to connect the body axes $Oxyz$ with the considered set of particles (Munk & McDonald 1960). From the point of view of simplifying the equations of motion, a convenient method is using the so–called *Tisserand systems* (Tisserand 1891).

To introduce Tisserand systems, let us write the absolute velocity (relative to $OXYZ$,) of a particle of $\mathcal{S}$ with position $\vec{x_i}$ and mass m_i as

$$\vec{V_i} = \vec{\omega} \times \vec{x_i} + \vec{v}_i(\vec{\omega}). \tag{1}$$

The vector $\vec{\omega}$ is common for the set $\mathcal{S}$ and, at this stage, arbitrary. In contrast $\vec{v}_i(\vec{\omega})$, the deformation or residual velocity (Moritz & Mueller 1987), depends on each particle i and the choice of $\vec{\omega}$.

Tisserand systems can be defined by any of the following conditions that fix $\vec{\omega}$ to a certain value $\vec{\omega_T}$:

(a) The angular momentum of $\mathcal{S}$

$$\vec{L} = \sum_{i \in \mathcal{S}} m_i \left(\vec{x_i} \times \vec{V_i} \right) \tag{2}$$

can be expressed as $\vec{L} = \mathbb{I}\,\vec{\omega_T}$ (Tisserand 1891), where $\mathbb{I}$ is the matrix of inertia of $\mathcal{S}$.

(b) The kinetic energy of $\mathcal{S}$ associated to the deformation velocity

$$\mathcal{T}_{\rm def}(\vec{\omega_T}) = \frac{1}{2} \sum_{i \in \mathcal{S}} m_i (\vec{v}_i(\vec{\omega_T}))^2 \tag{3}$$

is minimum (Jeffreys 1976).

(c) The *relative* angular momentum of $\mathcal{S}$ related with the deformation velocity

$$\vec{h}(\vec{\omega_T}) = \sum_{i \in \mathcal{S}} m_i \left[\vec{x}_i \times \vec{v}_i(\vec{\omega_T}) \right] \tag{4}$$

is the null vector (Tisserand 1891).

The former characterizations turn out to be equivalent, that is to say, (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a). The second and third implications are detailed, to some extent, in the existing literature (e.g., Moritz & Mueller 1987). Let us focus on the first one.

From Eqs. (1) and (3), the deformation kinetic energy can be written as (Escapa 2011)

$$\mathcal{T}_{\rm def}\left(\vec{\omega}\right) = \mathcal{T} - \vec{L}\vec{\omega} \ + \frac{1}{2}\vec{\omega} \,\mathsf{I}\, \vec{\omega}, \tag{5}$$

where $\mathcal{T}$ is the kinetic energy of $\mathcal{S}$. Hence, for an arbitrary vector $\vec{\lambda}$ different from $\vec{0}$, we have

$$\mathcal{T}_{\rm def}\left(\vec{\omega} + \vec{\lambda}\right) = \mathcal{T}_{\rm def}\left(\vec{\omega}\right) - \vec{L}\ \vec{\lambda} + \vec{\lambda}\,\mathsf{I}\,\vec{\omega} + \frac{1}{2}\vec{\lambda}\,\mathsf{I}\,\vec{\lambda}. \tag{6}$$

If we consider condition (a), defining the angular momentum of the system $\vec{L}$, in Eq. (6), we get

$$\mathcal{T}_{\rm def}\left(\vec{\omega}_T + \vec{\lambda}\right) - \mathcal{T}_{\rm def}\left(\vec{\omega}_T\right) = \frac{1}{2}\vec{\lambda}\,\mathsf{I}\,\vec{\lambda}. \tag{7}$$

Since the matrix of inertia is definite positive, we have that

$$\frac{1}{2}\vec{\lambda}\,\mathsf{I}\,\vec{\lambda} > 0,\ \ \vec{\lambda} \in \mathbb{R}^3,\ \ \vec{\lambda} \neq \vec{0}. \tag{8}$$

Therefore, Eq. (7) implies that $\mathcal{T}_{\rm def}\left(\vec{\omega}\right)$ takes its minimum at $\vec{\omega}_T$, i.e., condition (b).

2. TIME EVOLUTION OF TISSERAND SYSTEMS

The angular velocity $\vec{\omega}_T$, considered as a known function of time, determines the rotational kinematics of the body axes, but not its orientation in a univocal manner (Tisserand 1891). Specifically, from the components of $\vec{\omega}_T$ in the $OXYZ$ system, we can construct the skew–symmetric matrix

$$\Sigma_T(t) = \begin{pmatrix} 0 & -\omega_{TZ}(t) & \omega_{TY}(t) \\ \omega_{TZ}(t) & 0 & -\omega_{TX}(t) \\ -\omega_{TY}(t) & \omega_{TX}(t) & 0 \end{pmatrix}. \tag{9}$$

It allows defining a rotation matrix $\mathsf{R}(t)$ that brings the $OXYZ$ system to the body axes through (Wintner 1941)

$$\Sigma_T(t) = \frac{d\mathsf{R}^t}{dt}\mathsf{R}, \tag{10}$$

where the superscript t denotes the transpose of a matrix. The solution of this linear differential equation is given by

$$\mathsf{R}(t) = \mathsf{R}\left(t_0\right) \exp\left(-\int_{t_0}^{t} \Sigma_T(s) ds\right), \tag{11}$$

$\mathsf{R}\left(t_0\right)$ providing the numerical value of $\mathsf{R}(t)$ at the epoch t_0.

In this way, besides any of the conditions (a), (b), or (c), the specification of a particular Tisserand system requires providing explicitly the initial orientation of the body axes relative to $OXYZ$.

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3. REFERENCES

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