EFFECTS OF THE TIDAL MASS REDISTRIBUTION ON THE EARTH ROTATION

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ABSTRACT. The effects of the tidal mass redistributions on the Earth precession and nutations are revisited, under various hypothesis on the elastic response of the Earth and using the Hamiltonian approach. New non-negligible secular and periodic contributions have been found.

1. INTRODUCTION

The gravitational action of the Moon and the Sun on the deformable Earth perturbs its state by inducing in it a mass redistribution. In turn, such mass redistribution produces a variation of the gravitational energy of the system, leading to an additional term commonly referred to as *redistribution tidal potential*. In this regard, Moon and Sun are viewed as perturbed bodies. The effects of that redistribution potential on the forced rotational motion of the Earth figure axis have been previously discussed by Souchay and Folgueira (2000), Escapa et al. (2004), Ferrándiz et al. (2012) and Baenas (2014), within a Hamiltonian framework. Another approach to the problem, based in the SOS equations (Sasao et al. 1980), can be found in Lambert and Mathews (2006).

The Hamiltonian treatment of the elastic Earth follows the classic ideas by Love (1911) and assumes that the variation of the Earth's gravitational potential due to its tidal mass redistribution is proportional to the perturbing potential – Getino and Ferrándiz (1990, 1991, 1995), Kubo (1991), Escapa (2011). However, that proportionality can be modeled in various ways, adapted to different rheological hypothesis and different levels of mathematical complexity. A first, simplified model consists in considering a sole, global constant, within the Love's number approach (Munk and MacDonald 1960), to determine the additional gravitational potential at the deformed Earth surface. Besides, this simplified elastic behaviour has been profusely used to search the effects of the associated changes of the inertia tensor and kinetic energy on the Earth's rotation – which are indeed larger than those due to the incremental potential. However, it is only compatible with a rheological Earth model which is also simplified, the non-perturbed state being a non-rotating sphere (Wahr 1981).

Before introducing a more general elastic response in the analytical modeling, a rheological model based in Wahr (1981) can be considered as a first step. In such a situation closer to reality, the non-perturbed state is assumed to be ellipsoidal and rotating. The Earth's elastic response, seen, e.g., in the redistribution tidal potential, is described by means of a set of Love's numbers, which can depend on the order m of the spherical harmonics in the geopotential expansion and on the excitation frequencies as well. They form a set of complex numbers in the general case corresponding to some anelastic behaviour in the response, a case included, e.g., in the IERS Conventions 2010 (Petit and Luzum 2010). We denote those numbers by

$$\bar{k}_{2m} = \left|\bar{k}_{2m}\right| e^{i\varepsilon_{2m}}. \qquad (1)$$

From a dynamical point of view, that hypothesis requires an *ab initio* reconstruction of the rotation theory (Baenas 2014), in which the expression of the redistribution energy potential is given by the sum (over p and q, both representing either Moon or Sun) of terms of the form

$$\begin{aligned} V_{t;p,q} &= \frac{a_E^5}{r^3 r'^3} G m_p m_q \Big\{ \left|\bar{k}_{20}\right| \cos\varepsilon_{20} \mathscr{C}_{20}(\eta,\alpha)\, \mathscr{C}'_{20}(\eta',\alpha') \\ &+ \left|\bar{k}_{21}\right| \frac{1}{3} \left[\mathscr{C}_{21}(\eta,\alpha)\, \mathscr{C}'_{21}(\eta',\alpha'-\varepsilon_{21}) + \mathscr{S}_{21}(\eta,\alpha)\, \mathscr{S}'_{21}(\eta',\alpha'-\varepsilon_{21})\right] \\ &+ \left|\bar{k}_{22}\right| \frac{1}{12} \left[\mathscr{C}_{22}(\eta,\alpha)\, \mathscr{C}'_{22}\left(\eta',\alpha'-\frac{\varepsilon_{22}}{2}\right) + \mathscr{S}_{22}(\eta,\alpha)\, \mathscr{S}'_{22}\left(\eta',\alpha'-\frac{\varepsilon_{22}}{2}\right)\right]\Big\}, \end{aligned} \quad (2)$$

where $\mathscr{C}_{2j}$, $\mathscr{S}_{2j}$ and $\mathscr{C}'_{2j}$, $\mathscr{S}'_{2j}$, stand for the second degree real surface spherical harmonics, related to perturbed bodies (unmarked) and perturbing ones (with $'$) respectively, and relative to the terrestrial frame, (r,η,α) being the spherical coordinates – radial distance, colatitude and longitude. The symbol G denotes the gravitational constant; a_E is a conventional mean Earth's radius and m_p, m_q stands for the masses.

2. ANALYTICAL MODELING

An Andoyer-like set of canonical variables is used to describe the rotation linking the non-rotating system $OXYZ$ (an ecliptic frame) and the terrestrial one $Oxyz$ (a Tisserand mean system, Munk and MacDonald 1960, Escapa et al. this vol.), where O represents the Earth's barycenter. The canonical coordinates and conjugated momenta are denoted by $p = (\lambda,\mu,\nu)$, $q = (\Lambda, M, N)$, where M is the angular momentum modulus and Λ and N its projections on to the Z and z axes, respectively. The spherical harmonics in (2) must be expressed in terms of the spherical harmonics referred to the $OXYZ$ system, in which the orbital motions of Moon and Sun are provided by convenient ephemeris. The final expansion takes the form of a so-called *Poisson series* depending on the Andoyer variables and the fundamental arguments of nutation, denoted by (Kinoshita 1977)

$$\Theta_j = m_{1j} l + m_{2j} l' + m_{3j} F + m_{4j} D + m_{5j}\Omega. \quad (3)$$

Here l, l', F, D and Ω are the Delaunay variables of Moon and Sun. The subindex j stands for the 5-tuple of integers m_{ij}, so it can be used to indicate the functional dependence of $n_j = d\Theta_j/dt$. The coordinate λ and the auxiliary angle I (defined by $\cos I = \Lambda/M$) describe the motion of the Earth's angular momentum axis in the space system. The figure axis motion is given by the Euler's angles ψ, θ (longitude and obliquity), which are related to the Andoyer variables by the expansions (Kinoshita 1977)

$$\psi = \lambda + \sigma\frac{\sin\mu}{\sin I} + O(\sigma^2), \quad \theta = I + \sigma\cos\mu + O(\sigma^2), \quad (4)$$

which are accurate enough since the auxiliary angle σ (defined by $\cos\sigma = N/M$) has a magnitude about 10^{-6} rad, of the order of polar motion (Kinoshita 1977).

The Lie-Hori canonical perturbation method (Hori 1966) is used to tackle the evolution of the system with Hamiltonian $H = H_0 + H_1$, where the unperturbed part, $H_0 = T_0$, is the kinetic energy for a non-spherical symmetric rigid Earth (Kinoshita 1977) and the perturbed one is $H_1 = T_t + V_t$, in which T_t stands for the redistribution kinetic energy (Kubo 1991, Getino and Ferrándiz 1990, 1995) and V_t for the redistribution potential energy (2). Due to the linearity of the perturbation equations at the first order, the effects of T_t and V_t can be studied separately, and analytical expressions can be obtained for each component of the rotational motion of the Earth's figure axis (Baenas 2014).

The contribution of the mass redistribution to the precessional motion, denoted by δn_λ and δn_I, comes from the additional secular component of the Hamiltonian and can be determined from the variation of the velocities $n^*_\lambda = d\lambda^*/dt$ and $n^*_I = dI^*/dt$. Similar additive terms for the nutations, $\triangle\psi$ and $\triangle\theta$ are obtained taking into account (4) and the perturbation equations.

The solution to the precession rates caused by the Earth's mass redistribution can be expressed as

$$\begin{aligned} \delta n_\lambda &= -\frac{1}{\sin I^*}\frac{1}{CH_d}\sum_{p,q}^{M,S}\sum_{\substack{i,j\\ \tau\Theta_i-\epsilon\Theta_j=0}}\sum_{\tau,\epsilon}^{\pm 1}\sum_{m}^{0,1,2} \left|\bar{\kappa}_{2m,j;p}\right| k_q T^{(n_\lambda)}_{ijpq,m}(\tau,\epsilon)\cos\varepsilon_{2m,j}, \\ \delta n_I &= -\frac{1}{\sin I^*}\frac{1}{CH_d}\sum_{p,q}^{M,S}\sum_{\substack{i,j\\ \tau\Theta_i-\epsilon\Theta_j=0}}\sum_{\tau,\epsilon}^{\pm 1}\sum_{m}^{0,1,2} \left|\bar{\kappa}_{2m,j;p}\right| k_q T^{(n_I)}_{ijpq,m}(\tau,\epsilon)\sin\varepsilon_{2m,j}, \end{aligned} \quad (5)$$

where the functions $T^{(-)}_{ijpq,m}(\tau,\epsilon)$ depend on the auxiliary variable I and on the orbital solutions through the Kinoshita's (1977) B_i, C_i and D_i functions and are given by

$$\begin{aligned} T^{(n_\lambda)}_{ijpq,m}(\tau,\epsilon) &= \frac{9}{4}\frac{\partial B_{i;p}}{\partial I}B_{j;q}\delta_{m0} + 3\frac{\partial C_{i;p}}{\partial I}C_{j;q}\delta_{m1} + \frac{3}{4}\frac{\partial D_{i;p}}{\partial I^*}D_{j;q}\delta_{m2}, \\ T^{(n_I)}_{ijpq,m}(\tau,\epsilon) &= \tau m_{5i}\left(\frac{9}{4}B_{i;p}B_{j;q}\delta_{m0} - 3C_{i;p}C_{j;q}\delta_{m1} - \frac{3}{4}D_{i;p}D_{j;q}\delta_{m2}\right). \end{aligned} \tag{6}$$

The complex parameter $\bar{\kappa}_{2m,j;p}$ is a generalization of the constant defined by Kubo (1991)

$$\bar{\kappa}_{2m,j;p} = \frac{1}{3}\bar{k}_{2m;j} m_p a_E^2 \left(\frac{a_E}{a_p}\right)^3,$$

where subindex j points to the dependence on the orbital (or excitation) frequencies n_j. The constant k_q is the one defined by Kinoshita (1977), $H_d = 1 - A/C$ is the Earth's dynamical ellipticity, A and C being the equatorial and polar Earth's principal moments of inertia, and δ_{mk} is the Kronecker delta.

These analytical formulas show that the nonzero contribution to the precessional rate in obliquity, δn_I, is a purely anelastic effect, as it only stands for complex values of the Love's numbers (with any $\varepsilon_{2m,j} \neq 0$), what is in accordance with Lambert and Mathews (2006).

3. RESULTS

It can be shown analytically (Escapa et al. 2004, Baenas 2014) that in the case of the simplified Earth's elastic response, with $\bar{k}_{2m} = k \in \mathbb{R}$, the effects of the different harmonic contributions of the redistribution potential cancel each other out in all cases: precession velocities and nutation terms. When more general rheological models for the Earth's mantle elasticity are considered, there appear non-negligible secular and periodic contributions to the motion of the Earth's figure axis.

For the evaluation of the analytical solutions, the frequency dependent complex Love's numbers have been taken from IERS Conventions 2010. Table 1 shows the results for the contributions to the precession rates, including separately the additive terms coming from the well-known harmonic contributions of the perturbing tidal potential: zonal, tesseral and sectorial, denoted respectively by B, C and D. In the zonal part, the permanent tide contribution, B_0, is computed separately. This particular term must be included or removed, depending on the dynamical model considered for the rigid part of the Earth's inertia tensor ("zero tide" or a "tide free" according to IERS Conventions 2010 terminology).

	Zonal		Tesseral	Sectorial	Total
	B_0	$B - B_0$	C	D	
δn_λ	43.7900	−4.1389	−60.6554	27.0102	6.0059
δn_I	0.0000	-0.0118	0.1209	0.6656	0.7748

Table 1: Contribution of the mass redistribution to the precessional rates (unit 1 mas/cJ).

Table 2 displays only the in-phase amplitudes of the main nutation terms. They are computed from analytical expressions that extend (5) and correspond to the non-vanishing combinations $\tau\Theta_i - \epsilon\Theta_j$ ($\tau, \epsilon = \pm 1$) of the fundamental arguments of nutation (3), where Θ_i stands for the perturbed bodies and Θ_j for the perturbing ones. For the sake of briefness, the contributions B_0, $B - B_0$, C and D have not been shown separately in Table 2. The out-of-phase contributions are smaller in magnitude.

The numerical results show a significant influence of the frequency dependence of the Love's numbers. This effect is mainly due to the existence of the free core nutation (FCN) resonance processes in the diurnal band.

Considering the complete mass redistribution contribution, kinetic and potential energies, the differences with respect to the simplified elastic model reach significant values: about 6 mas/cJ for the velocity of precession in longitude, 0.8 mas/cJ for the velocity of precession in obliquity, 140 μas in the amplitude of the nutation in longitude with period of 13.66 days, and 50 μas in the amplitude of the nutation in obliquity for the same component.

Finally it can be noted that the analytical formulation allows the inclusion of different rheological models, which can be considered as a numerical input for the rotation solution, in a similar way than the orbital motion of the perturbing bodies.

Argument					Period	T_t		V_t	
l	l'	F	D	Ω	days	$\Delta\psi$	$\Delta\theta$	$\Delta\psi$	$\Delta\theta$
+0	+0	+0	+0	+1	−6793.48	+933.35	−274.99	+5.4095	−11.5748
+0	+0	+0	+0	+2	−3396.74	−18.00	+6.55	−1.0599	+0.5897
+0	+1	+0	+0	+0	365.26	−43.06	−58.19	+0.1294	−0.1903
+0	−1	+2	−2	+2	365.25	+19.71	−6.69	+0.0025	−0.0033
+0	+0	+2	−2	+2	182.63	−2338.50	+844.09	+1.8798	−0.9666
+0	+1	+2	−2	+2	121.75	−138.48	+50.26	+0.0282	−0.0150
+1	+0	+0	+0	+0	27.55	+21.65	−311.39	+0.0938	+0.0037
+0	+0	+2	+0	+2	13.66	−5537.17	+2043.26	−0.2686	+0.1373
+0	+0	+2	+0	+1	13.63	−1134.83	+349.43	−0.0027	+0.0157
+1	+0	+2	+0	+2	9.13	−1101.47	+408.67	+0.0372	−0.0186

Table 2: Contribution of the mass redistribution to the figure axis nutations (unit 1 μas).

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