ON SOLUTION OF SECULAR SYSTEM IN THE ANALYTICAL MOON'S THEORY

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The theory of the orbital motion of the Moon in the frameworks of the general planetary theory GPT (Brumberg, 1995) enables to represent the lunar coordinates in a purely trigonometric form and valid, at least formally, for an indefinite interval of time. For that the Moon is considered to be an additional planet in the field of eight major planets. The trigonometric form of the coordinates is ensured by a special technique of the solution of the lunar equations that enables to separate the short–period and long–period terms arguments. The long–period terms form an autonomous secular system. The trigonometric solution of this system describes the secular motions of the lunar perigee and node with taking into account the secular planetary inequalities. The secular system in Laplace–type variables was constructed in (Ivanova, 2013). The aim of this paper is to solve this secular system by the normalizing Birkhoff transformation.

The basic series of the Moon's theory have the form

$$p = \sum_{m=0}^{\infty} \sum_{i+j+k+l=m} p_{i,j,k,l}(t) \prod_{n=1}^{9} a_n^{i_n} \overline{a}_n^{k_n} b_n^{l_n} \overline{b}_n^{m_n} \,, \quad w = \sum_{m=1}^{\infty} \sum_{i+j+k+l=m} w_{i,j,k,l}(t) \prod_{n=1}^{9} a_n^{i_n} \overline{a}_n^{k_n} b_n^{l_n} \overline{b}_n^{m_n} \tag{1}$$

where the dimensionless complex conjugate variables p, q and real variable w representing small deviations from the planar circular motion are introduced instead of geocentric lunar rectangular coordinates. a_n, b_n are the complex Laplace–type variables proportional to the eccentricity and inclination of the body with number n. The coefficients in (1) are quasi–periodic functions of mean longitudes of the major planets ($\lambda_i, i = 1, 2, \ldots, 8$) and the Moon ($\lambda_9$)

$$p_{i,j,k,l}(t) = \sum_{\gamma} p_{i,j,k,l,\gamma} \exp \sqrt{-1}\,(\gamma\lambda)\,, \qquad w_{i,j,k,l}(t) = \sum_{\gamma} w_{i,j,k,l,\gamma} \exp \sqrt{-1}\,(\gamma\lambda)\,, \tag{2}$$

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_9), \quad (\gamma\lambda) = \sum_{i=1}^{9} \gamma_i \lambda_i, \quad \sum_{i=1}^{9} \gamma_i = 0.$$

The coefficients $p_{i,j,k,l,\gamma}$ and $w_{i,j,k,l,\gamma}$ are the functions of the semi–major axes, mean motions and masses of the bodies under consideration. It should be noted that the planetary theory is limited by the keplerian terms, the first–order with respect to the masses intermediary and linear theory. It is sufficient for constructing the theory of motion of the Moon. The terms with $m = 0$ in (1) represent an intermediate solution independent of eccentricities and inclinations of all the bodies. It is sought by iterations. In the previous papers (for instance, Ivanova, 2013) it had the form of the Poisson series due to the expansion of the frequency denominators with respect to the ratios of the mean motions of the planets and the Moon. In this paper the intermediate solution has the form of the echeloned series (Ivanova, 2001) without any expansion. The terms with $m > 0$ in (1) are obtained for non–zero values of the eccentricities and inclinations of all the bodies. The technique of Birkhoff normalization for separating fast and slowly changing variables is used here. The terms which do not enable to be integrated without secular terms correspond to critical combinations of multi–indices satisfying the relations

$$\sum_{n=1}^{9} i_n - j_n + k_n - l_n = 1, \qquad \gamma_n = \delta_{9,n} - i_n + j_n - k_n + l_n \,, \tag{3}$$

$\delta_{i,n}$ being the Kronecker symbol. They are represented in the similar manner as the series (1) with (2). Such terms form an autonomous secular system

$$\dot{\alpha} = \sqrt{-1}\, N \left[A\alpha + \Phi(\alpha, \overline{\alpha}, \beta, \overline{\beta}) \right] , \quad \dot{\beta} = \sqrt{-1}\, N \left[B\beta + \Psi(\alpha, \overline{\alpha}, \beta, \overline{\beta}) \right] \tag{4}$$

in slowly changing variables $\alpha = (\alpha_1, \ldots, \alpha_9)$ $(\alpha_i = a_i \exp -\sqrt{-1}\lambda_i)$ and $\beta = (\beta_1, \ldots, \beta_9)$ $(\beta_i = b_i \exp -\sqrt{-1}\lambda_i)$. To complete this system one should add the corresponding conjugate equations. Here N is the diagonal matrix of mean motions of the planets and the Moon, A and B are 9×9 constant matrices of semi-major axes, mean motions and masses of all the bodies under consideration. 9–vectors Φ, Ψ are represented by the power series with quasi–periodic coefficients

$$\Phi = \sum{}^{*}\, \Phi_{i,j,k,l,\gamma} \prod_{n=1}^{9} \alpha_n^{i_n} \overline{\alpha}_n^{k_n} \beta_n^{l_n} \overline{\beta}_n^{m_n} , \quad \Psi = \sum{}^{*}\, \Psi_{i,j,k,l,\gamma} \prod_{n=1}^{9} \alpha_n^{i_n} \overline{\alpha}_n^{k_n} \beta_n^{l_n} \overline{\beta}_n^{m_n} . \tag{5}$$

Functions Φ, Ψ contain only forms of odd degree in slowly changing variables α, $\overline{\alpha}$, β, $\overline{\beta}$ starting with the third degree terms. The asterisk at the summation sign indicates that this summation is taken only over critical values (3). Let μ_j, $S_{.j}$ and ν_j, $T_{.j}$ $(j = 1, 2, \ldots, 9)$ be the eigenvalues and eigenvectors of the matrices NA and NB in (4), respectively. Then the first step to solve the secular system (4) is the construction of the linear transformations $\alpha = Sx\,, \quad \beta = Ty$ which transform (4) into the form

$$\dot{X} = \imath \big[PX + NR(X) \big] \tag{6}$$

with 9–vectors of new variables $X = (x, \overline{x}, y, \overline{y})$ $(x, \overline{x}, y, \overline{y}$ are 9–vectors) and new right–hand members presented by 9–vectors $R_1 = N^{-1}S^{-1}N\Phi\,, \quad R_2 = -\overline{R}_1\,, \quad R_3 = N^{-1}T^{-1}N\Psi\,, \quad R_4 = -\overline{R}_3\,.$ P is 9×9 diagonal matrix of the structure: $P = diag(\mu\,, -\mu\,, \nu\,, -\nu)$. The resulting secular system (6) is solved by Birkhoff normalization technique which consists of constructing the formal power series $X = Y + \Gamma(Y)$ with new variables $Y(u, \overline{u}, v, \overline{v})$ $(u, \overline{u}, v, \overline{v}$ are 9–vectors) reducing (6) to the system

$$\dot{Y} = \imath \big[PY + NF(Y) \big] \tag{7}$$

with the power series F admitting the straightforward integration of this system without t–secular terms. Functions Γ and F representing the series in powers of Y are found by iteration technique. Since the influence of the Moon on the major planets is not taken into account the system (7) splits into two secular systems for the Moon and for the planets. The last of them is determined in GPT. As a result, the equations (7) for the Moon are transformed into the system of linear equations

$$\dot{u}_9 = \sqrt{-1}\, u_9\, (\mu_9 + \delta\mu_9), \qquad \dot{v}_9 = \sqrt{-1}\, v_9\, (\nu_9 + \delta\nu_9)\,. \tag{8}$$

$\delta\mu_9$ and $\delta\nu_9$ are real constant corrections to corresponding eigenvalues for the Moon. Hence, these equations admit the straightforward integration and the solution has the form

$$u = \eta_{1,9}\, \exp \sqrt{-1}\, \varphi_{1,9}, \qquad v = \eta_{2,9}\, \exp \sqrt{-1}\, \varphi_{2,9}, \tag{9}$$

$$\begin{pmatrix} \varphi_{1,9} \\ \varphi_{2,9} \end{pmatrix} = \begin{pmatrix} \mu_9 + \delta\mu_9 \\ \nu_9 + \delta\nu_9 \end{pmatrix} t + \begin{pmatrix} \tau_{1,9} \\ \tau_{2,9} \end{pmatrix}, \qquad \begin{pmatrix} \delta\mu_9 \\ \delta\nu_9 \end{pmatrix} = n \sum{}^{*} \begin{pmatrix} U^1_{km} \\ U^2_{km} \end{pmatrix} \prod_{i=1}^{9} \eta_{1,i}^{2k_i} \eta_{2,i}^{2m_i}$$

with real constants of integration $\eta_{j,i}$ and $\tau_{j,i}$ $(j = 1, 2\,;\, i = 1, \ldots, 9)$.

The resulting solution of the secular system for the Moon has the form

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \sum \begin{pmatrix} \alpha_{klmn} \\ \beta_{klmn} \end{pmatrix} \prod_{i=1}^{9} \exp \sqrt{-1} \Big[(k_i - l_i)\, \varphi_{1,i} + (m_i - n_i)\, \varphi_{2,i} \Big] , \quad \sum_{i=1}^{9} (k_i - l_i + m_i - n_i) = 1\,. \tag{10}$$

The trigonometric solution of the secular system has the semi–analytical form with numerical coefficients α_{klmn} and β_{klmn}. It includes terms due to the secular evolution of the lunar perigee and node as well as of that of the major planets.

REFERENCES

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