A GENERALIZED THEORY OF THE FIGURE OF THE EARTH: APPLICATION TO THE MOMENT OF INERTIA AND GLOBAL DYNAMICAL FLATTENING

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ABSTRACT. A new integrated formula to obtain the equilibrium figures interior the earth to third-order accuracy is developed. In this formula, both the direct and indirect contribution of the anti-symmetric crust layer are included, as result, all the non-zero order and odd degree terms are included in the spherical harmonic expression of the equilibrium figures.

The moments of inertia (MoI: A,B,C) and global dynamic flattening (H) are important quantities in research of rotating Earth. Accurate precession and gravity observations give $H_{obs} \approx 1/305.5$, while its value from rotating symmetric PREM model and traditional theories of equilibrium figures, H_{PREM}, is about 1.1% less than H_{obs}.

Using our new potential theory and replacing the homogenous outermost crust and oceanic layers in PREM with CRUST2.0 model data, we recalculate the geometrical flattening profile of the Earth interior and finally get the values of MoI and H. Their consistencies with observations are significantly improved.

1. A GENERALIZED THEORY OF THE FIGURE OF THE EARTH

If the isotropic Earth is in hydro-static-equilibrium (HSE) and rotating constantly, the interior surfaces of equi-density, of equi-potential and of equi-pressure will coincide with each other. Let r denote the distance between a point in the Earth and the geocenter, and s is the mean equi-volumetric radii of the equi-potential surface cross this point. The traditional expression of r is:

$$r(s,\theta) = s\left[1 + \sum_{n=0}^{\infty} s_{2n}(s) P_{2n}(\cos\theta)\right] \tag{1}$$

The gravity (gravitational plus centrifugal) potential at this point can be expressed as:

$$W(s,\theta) = \frac{4}{3}\pi G\bar{\rho}s^2 \sum_{n=0}^{\infty} F_{2n}(s) P_{2n}(\cos\theta) \tag{2}$$

where $F_{2n}(s)$ is function of $s_{2n}(s)$, and $\bar{\rho}$ is the mean density of the whole Earth.
Because W is a constant and does not depend on the colatitude θ on any given level surface, then $F_{2n} = 0\,(n \neq 0)$. If truncated at $n = 1$, 2 or 3, it degenerates respectively to Clairaut equation (first-order accuracy), Darwin - de Sitter equation (second-order accuracy), and *Denis'* formula(*Denis*, 1989) (third-order accuracy). In eq(1), there is no term of longitude ϕ nor of odd degree P_{2n+1}, meaning that the equilibrium figures of the Earth must be rotational symmetric and equatorial (south-north) symmetric, and their details can be found in *Moritz (1990)*, *Denis (1989)* and *Denis et al. (1997).* However, our real earth is not of so beautiful symmetric, instead, of topography and the geoid is also non-symmetric. In order to calculate, in more general, the figures of internal equi-potential surface and the geoid which are non-symmetric (Figure 1), we should replace eq.(1) by following equation:

$$r(s,\theta,\phi) = s\left[1 + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} H_n^m(s) Y_n^m(\theta,\phi)\right], \tag{3}$$

All the non-zero order and odd degree terms are included in the above spherical harmonic expression of the equilibrium figures. N has to be truncated in practice and it takes 6 in this work.

From the figures of internal equi-density surfaces given by above, i.e., the density distribution inside the whole earth is known, the moments of inertia (MOI: A,B,C) and global dynamic flattening $H = \frac{(C-(A+B)/2)}{C}$ can then be calculated easily.

We consider the earth model that is the same of PREM (*Dziewonski & Anderson, 1981*), but the curst layer of depth of 71km in PREM is replaced by CRUST2.0 model (*Chulick et al., 2002*; *Mooney et al., 1998*) that consists of 7 different layers of depth up to 70.137 km and of $2^o * 2^o$ grid. One constrain is required here that the total mass of the earth be conserved. And the most part under the 71 km depth is still assumed to be in HSE. The inhomogeneous crust layer considered here is shown in following skeptical Figure 2.

Two effects of the loading of the anti-symmetric crust layer are considered here:

1) direct effect: the crust inhomogeneous mass change directly the gravitational potential for all mass points interior in different ways, therefore, the figures of equi-potential surfaces interior are changed without symmetries;

2) indirect effect: As the figures of equi-density surfaces (then the density distribution) interior are changed by the direct effect, the gravitational potential of other locations (outside/inside this surface) are changed, and the figures of equi-potential surfaces all through the earth are then changed again. This process is reciprocal and needs iteration, and will finally reach equilibrium. And all the figures can be gotten in more general, i.e., without any symmetry (Figure 3).

Incorporating both the direct effect and indirect effect, the gravity potential $W(s)$ crossing the point $r(s, \theta, \phi)$ becomes

$$\begin{aligned} W &= V_{in} + V_{out} + Z \\ &= GE_0(s) + G\bar{\rho}s^2 \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_n^m \left[m_h p_{n,m} + \sum_{l=0}^{\infty} \frac{s^{l-2}}{\bar{\rho}} u_{l,n,m} + \sum_{l=1}^{\infty} g_{l,n,m} + \sum_{l=0}^{\infty} f_{l,n,m} \right] \\ &= GE_0(s) + G\bar{\rho}s^2 \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_n^m(\theta,\phi)\Xi_n^m(s), \end{aligned} \tag{4}$$

where

$$\Xi_n^m = m_h p_{n,m} + \sum_{l=0}^{\infty} \frac{s^{l-2}}{\bar{\rho}} u_{l,n,m} + \sum_{l=1}^{\infty} g_{l,n,m} + \sum_{l=0}^{\infty} f_{l,n,m} \tag{5}$$

where, $p_{n,m}, u_{l,n,m}, g_{l,n,m}$ and $f_{l,n,m}$ are complex functions of all $H_n^m(s_i)$ (not only the figure coefficients of the surface of equivalent radii s, but also the figure coefficients of all other surfaces of equivalent radii s_i). And the final equations of the figure coefficients $H_n^m(s_i)$ can be expressed in following form, with the requirement that the gravity potential $W(s)$ on the equilibrium surface should be independent on the the colatitude θ and longitude ϕ,

$$\begin{cases} \Xi_n^m + (-1)^m \Xi_n^{-m*} = 0 \\ n = 1, \ldots, \infty \\ m = 0, \ldots, n \end{cases} \tag{6}$$

In eqs.(6), there are $\sum_{n=1}^{N}(n+1) = \frac{1}{2}N(N+3)$ equations for each surface (or layer), while do not forget that the earth model is discreted by K layers and K is usually bigger than several hundreds.

The detail derivation of the formula, the procedure to solve the complex equation system of the spherical harmonics of the figures, and its validation by comparisons with other results when we degenerate this theory to that of *Denis* by keeping only H_2^0, H_4^0 and H_6^0, will be presented in another paper, and we present here only the primary results.

2. APPLICATION TO THE CALCULATION OF THE MOMENT OF INERTIA AND GLOBAL DYNAMICAL FLATTENING

The moments of inertia (MoI: A, B, C) and global dynamic flattening (H) are important quantities in research of rotating Earth. Very accurate precession and gravity observations give $H_{obs} \approx 1/305.5$, while

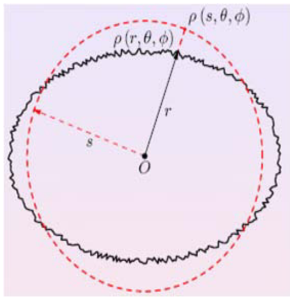

Figure 1: Expression of a general figure of equipotential surface interior the earth

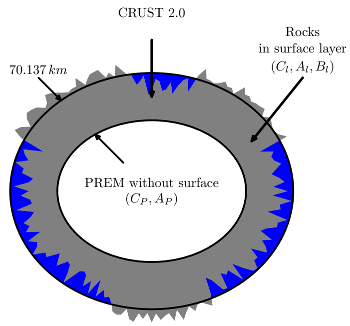

Figure 2: Skeptical earth model containing PREM and CRUST2.0

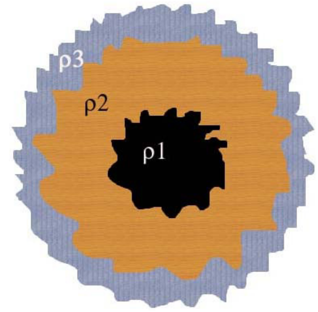

Figure 3: Skeptical plot of the change of all surfaces interior caused by both direct and indirect effects of the inhomogeneous crust

its value from rotating symmetric PREM model and traditional theories of equilibrium figures mentioned above, H_{PREM}, is about 1/308.5, approximately 1.1% less than H_{obs}. This phenemenoun and its possible interpretation have been discussed in several papers (e.g., *Defraigne (1997)*, *Dehant & Capitaine (1997)*, *Mound et al. (2003)*, etc.) with various kinds of assumptions, we will not discussed them here and readers are recommended to refer them for detail. In this paper, we skip these assumptions and tend to search reason from the theory of the earth figure itself.

The direct contribution of the crust layer to MoI and H are listed as following:

Table 1: Direct contribution of the crust layer to MoI and H

	A $10^{37}\ kg \cdot m^2$	B $10^{37}\ kg \cdot m^2$	C $10^{37}\ kg \cdot m^2$	1/H
PREM(-71km)	7.7087284	7.7087284	7.7336553	
CRUST2.0	0.2949340	0.2947971	0.2956929	
TOTAL	8.0036624	8.0035255	8.0293482	311.7674842

While, if considering both direct and indirect effect of the crust layer, the value of MoI and H are listed below:

Table 2: Direct & indirect contributions of the crust layer to MoI and H

	A $10^{37}\ kg \cdot m^2$	B $10^{37}\ kg \cdot m^2$	C $10^{37}\ kg \cdot m^2$	1/H
PREM(All)	8.0112987	8.0112987	8.0373506	308.5131401
PREM(-71km)	7.7164775	7.7164823	7.7418221	
CRUST2.0	0.2948918	0.2947790	0.2957497	
TOTAL	8.0113693	8.0112612	8.0375718	306.1164533

From above tables, if consider the direct effect of the curst layer only, H deviates from the H_{obs} more; but incorporating the indirect effect can reduce significantly the difference between H and H_{obs} from 1.1% to 0.2%, while we do not need any other assumptions in this work.

3. SHORT SUMMARY

In this short paper, the principle of a generalized theory to obtain the equilibrium figures interior the earth to fully third-order accuracy is presented. In this theory, both the direct and indirect contribution of the anti-symmetric crust layer are included, as result, all the non-zero order and odd degree terms are included in the spherical harmonic expression of the equilibrium figures.

As a application (as well as a validation) of this new theory, we recalculate the moments of inertia (MoI) and global dynamic flattening (H), and it shows that the consistencies of H with observations H_{obs} are significantly improved from 1.1% to 0.2%, without any other assumption.

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