# ON THE DEFINITION OF A REFERENCE FRAME AND THE ASSOCIATED SPACE IN A GENERAL SPACETIME

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ABSTRACT. A reference frame $\mathrm{F}$ can be defined as an equivalence class of spacetime charts (coordinate systems) having a common domain $\mathrm{U}$ and exchanging by a spatial coordinate change. The associated physical space is made of the world lines having constant space coordinates in any chart of the class. This is a local definition. The data of a global 4-velocity field $v$ defines a global "reference fluid". The associated global physical space is made of the maximal integral curves of that vector field. Assume that the local and global spaces correspond with the same three-dimensional network of observers. In that case, the local space can be identified with a part (an open subset) of the global space.

## 1. INTRODUCTION

A reference frame is essentially a three-dimensional network of observers equipped with clocks and meters. To any reference frame, one should be able to associate some three-dimensional *space,* in which the observers of the network are at rest (even though their mutual distances may depend on time). Clearly, both notions are fundamental ones for physics. In the relativistic theories of gravitation, the spacetime metric tensor $g_{\mu\nu}$ is a field, thus it depends in particular on time. Hence, one expects that relevant reference frames are not rigid. The relevant notion is that of a *reference fluid,* given by a 4-velocity field $v$ on spacetime: $v$ is the unit tangent vector field to the world lines of the observers belonging to the network (Cattaneo, 1958). In standard practice, one often admits implicitly that a reference frame can be fixed by the data of one *coordinate system* (or *chart*). The link with the definition by the 4-velocity field $v$ tangent to a network of observers was also given by Cattaneo (1958). Any admissible chart on the spacetime, $\chi : X \mapsto (x^\mu)$ $(\mu = 0, ..., 3)$, defines a unique network of observers, whose world lines are

$$x^j = \text{Constant } (j = 1, 2, 3),\ \ x^0 \text{ variable.} \tag{1}$$

The corresponding four-velocity field $v$ has the following components in the chart $\chi$:

$$v^0 \equiv \frac{1}{\sqrt{g_{00}}}, \qquad v^j = 0 \quad (j = 1, 2, 3). \qquad [\text{signature } (+ - - -)\,] \tag{2}$$

We note, however, that this is valid only within the domain of definition $\mathrm{U}$ of the chart $\chi$, thus in general not in the whole spacetime.

The notion of the space associated with a network of observers was missing in the general-relativistic literature. But in practice, one cannot dispense with some notion of a *physical space.* One needs to define the spatial positions of physical objects, even though these depend on the reference network considered. One also needs a physical space to define the quantum space of states, and spatial vectors or tensors such as the usual 3-velocity vector or the rotation rate tensor of a triad. We recall previous results (Arminjon & Reifler, 2011) that provide *local* definitions. Then we announce results of a current work, that aims at defining *global* notions and at relating them to the formerly introduced local notions.

## 2. LOCAL DEFINITION OF A REFERENCE FRAME AND ASSOCIATED SPACE

One may formally define a reference frame as being an *equivalence class of charts* which are all defined on a given open subset $\mathrm{U}$ of the spacetime $\mathrm{V}$ and are related two-by-two by a *purely spatial* coordinate change:

$$x'^0 = x^0, \quad x'^k = \phi^k((x^j)) \qquad (j, k = 1, 2, 3). \tag{3}$$

This does define an equivalence relation (Arminjon & Reifler, 2011). Thus a reference frame $\mathrm{F}$, i.e. an equivalence class for this relation, can indeed be given by *the data of one chart* $\chi : X \mapsto (x^\mu)$ *with its domain of definition* $\mathrm{U}$ (an open subset of the spacetime manifold $\mathrm{V}$). Namely, $\mathrm{F}$ is the equivalence class of $(\chi, \mathrm{U})$. I.e., $\mathrm{F}$ is the set of the charts $\chi'$ which are defined on $\mathrm{U}$, and which are such that the transition map $f \equiv \chi' \circ \chi^{-1} \equiv (\phi^\mu)$ corresponds with a purely spatial coordinate change (3). The local physical space $\mathrm{M} = \mathrm{M_F}$ is defined as the set of the world lines (1), which are implicitly restricted to the common domain $\mathrm{U}$ of the charts $\chi \in \mathrm{F}$. Consider any given chart $\chi \in \mathrm{F}$. With any world line $l \in \mathrm{M_F}$, let us associate the triplet $\mathbf{x} \equiv (x^j)$ made with the *constant* spatial coordinates of the points $X \in l$ in the chart $\chi$. We thus define a mapping

$$\widetilde{\chi} : \mathrm{M_F} \to \mathbb{R}^3, \quad l \mapsto \mathbf{x} \text{ such that } \forall X \in l,\ \chi^j(X) = x^j \ (j = 1, 2, 3). \tag{4}$$

We defined a structure of differentiable manifold on $\mathrm{M_F}$, for which the set of the mappings $\widetilde{\chi}$ (for $\chi \in \mathrm{F}$) is an atlas: *The spatial part of any chart* $\chi \in \mathrm{F}$ *defines a chart* $\widetilde{\chi}$ *on* $\mathrm{M_F}$ (Arminjon & Reifler, 2011).

## 3. THE GLOBAL SPACE ASSOCIATED WITH A TIME-LIKE VECTOR FIELD

Given a global vector field $v$ on the spacetime $\mathrm{V}$, and given an event $X \in \mathrm{V}$, let $C_X$ be the solution of

$$\frac{dC}{ds} = v(C(s)), \qquad C(0) = X \tag{5}$$

that is defined on the *largest possible* open interval $\mathrm{I}_X$ containing $0$ (Dieudonné, 1971). Call the *range* $l_X \equiv C_X(\mathrm{I}_X) \subset \mathrm{V}$ the "maximal integral curve at $X$". If $X' \in l_X$ , then it is easy to show that $l_{X'} = l_X$. The *global space* $\mathrm{N}_v$ *associated with the vector field* $v$ is the set of the maximal integral curves of $v$:

$$\mathrm{N}_v \equiv \{l_X;\ X \in \mathrm{V}\}. \tag{6}$$

A chart $\chi$ with domain $\mathrm{U} \subset \mathrm{V}$ is said "$v$–adapted" iff the spatial coordinates remain constant on any integral line $l$ of $v$ — more precisely, remain constant on $l \cap \mathrm{U}$: For any $l \in \mathrm{N}_v$, there is some $\mathbf{x} \equiv (x^j) \in \mathbb{R}^3$ such that

$$\forall X \in l \cap \mathrm{U}, \quad P_S(\chi(X)) = \mathbf{x}. \tag{7}$$

(Here $P_S : \mathbb{R}^4 \to \mathbb{R}^3, \quad \mathbf{X} \equiv (x^\mu) \mapsto \mathbf{x} \equiv (x^j)$ is the spatial projection.) For any $v$–adapted chart $\chi$, the mapping

$$\bar{\chi} : l \mapsto \mathbf{x} \text{ such that (7) is verified} \tag{8}$$

is well defined on

$$\mathrm{D_U} \equiv \{l \in \mathrm{N}_v;\ l \cap \mathrm{U} \neq \emptyset\}. \tag{9}$$

Call the $v$–adapted chart $\chi$ "nice" if the mapping $\bar{\chi}$ is one-to-one. Assume the global vector field $v$ on $\mathrm{V}$ is non-vanishing (which is true if $v$ is time-like) and "normal" (which means that the flow of the field $v$ is indeed non-pathological in some technical sense). Then, for any point $X \in \mathrm{V}$, there exists a nice $v$–adapted chart $\chi$ whose domain is a neighborhood of $X$. Consider the set $\mathcal{F}_v$ made of all nice $v$–adapted charts on the spacetime manifold $\mathrm{V}$. Define the set $\mathcal{A}$ made of the mappings $\bar{\chi}$, where $\chi \in \mathcal{F}_v$, Eq. (8). We can define a structure of differentiable manifold on $\mathrm{N}_v$, for which the set $\mathcal{A}$ is an atlas. The manifold structure gives a firm status to the space attached to a reference network and allows us to define spatial tensors naturally, as tensor fields on the space manifold.

Let $v$ be a normal non-vanishing vector field on $\mathrm{V}$, and let $\mathrm{F}$ be a reference frame as in Sect. 2, but *made of nice $v$–adapted charts,* all defined on the same open set $\mathrm{U} \subset \mathrm{V}$. We can show that the local space $\mathrm{M_F}$ is made of the intersections with the local domain $\mathrm{U}$ of the world lines belonging to $\mathrm{N}_v$. We may identify the local space $\mathrm{M_F}$ with the subset $\mathrm{D_U}$ of the global space $\mathrm{N}_v$, which is open in $\mathrm{N}_v$.

## 4. REFERENCES


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