AVERAGING IN THE N-BODY PROBLEM WITH THE LIE-SERIES METHOD IN STANDARD OSCULATING ELEMENTS

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ABSTRACT. A new method allowing to average the equations of motion in the N-body problem in a set of standard osculating elements with the usage of all the standard expansions of perturbing functions is proposed. The main idea is to double the number of variables and to conduct an averaging in a corresponding extended phase space. The additional variables disappear from the final results and transformation formulae. Once obtained, the averaged equations of motion can be applied in semi-analytical schemes for numerical integration as well as for further dynamical studies.

Only exceptional cases of celestial mechanical problems allow a construction of a precise analytical solution. As a rule, these are the model problems which allow to get an idea about the global behavior of a solution but are not precise enough to get the real positions of celestial objects. As an alternative, a series of methods to obtain an analytical solution in an approximate way has been elaborated (Le Verrier, Gauss, Lindstedt, Hamilton, Jacobi, Poincaré, Hori, Deprit and others) and successfully applied for solar system ephemerides, satellite and asteroid theories. Due to the presence of numerous resonances, not even a formal convergence can be guaranteed in most cases. The semi-analytical approach can then be applied based on the idea going back to Poincaré's statement that the short-periodic perturbations do not play a significant role in the long-term dynamics. The main problem of going beyond the first order approximation is not only the immense size of analytical calculations but also due to the fact that the differential equations of the N-body problem do not allow a canonical representation in standard osculating elements of the orbits. The two known possibilities to deal with a single Hamiltonian are:
1. The Jacobi Hamiltonian formalism (Jacobi, 1842), where the position and velocity of a planet m_1 are given in a reference frame with origin in m_0; position and velocity of m_2 are given in a reference frame with origin in the barycenter of m_1 and m_0, etc. Due to this hierarchical structure, no general expression for the perturbing function exists.
2. The Poincaré reduction (Laskar & Robutel, 1995) where the angular variables and the corresponding momenta are defined in two different reference systems. This approach necessarily introduces non-osculating elements in the expansions. Both of these methods require the introduction of generalized orbital elements instead of standard osculating elements (Beaugé et al., 2007). We propose here a way to use the advantages of a canonical formalism in standard osculating elements. To conduct the transformation, the Lie-series method has been chosen. This method gives the solution and the transformation formulae in explicit form and (in contrast to Poincaré's method) avoids the appearance of mixed terms.

In fact, every differential equation can be written in canonical form with the help of some additional variables. To give an idea, let us consider the equations of motion of two bodies a and b around a central body in some system of osculating canonical elements

$$\frac{\partial x_i^a}{\partial t} = \frac{\partial R^a}{\partial x_{i+3}^a}, \quad \frac{\partial x_{i+3}^a}{\partial t} = -\frac{\partial R^a}{\partial x_i^a}; \quad \frac{\partial x_i^b}{\partial t} = \frac{\partial R^b}{\partial x_{i+3}^b}, \quad \frac{\partial x_{i+3}^b}{\partial t} = -\frac{\partial R^b}{\partial x_i^b}. \tag{1}$$

These equations can be written as one canonical system after introducing additional variables $\mathbf{Y} = (\mathbf{y}^a, \mathbf{y}^b)$ conjugate to $\mathbf{X} = (\mathbf{x}^a, \mathbf{x}^b)$ in the $2 * 2 * 6$ dimensional phase space as

$$\frac{\partial \mathbf{X}}{\partial t} = \frac{\partial F}{\partial \mathbf{Y}}, \quad \frac{\partial \mathbf{Y}}{\partial t} = -\frac{\partial F}{\partial \mathbf{X}}, \tag{2}$$

with the Hamiltonian

$$F = \sum_{i=1}^{3} \left(y_i^a \frac{\partial}{\partial x_{i+3}^a} - y_{i+3}^a \frac{\partial}{\partial x_i^a} \right) R^a + \sum_{i=1}^{3} \left(y_i^b \frac{\partial}{\partial x_{i+3}^b} - y_{i+3}^b \frac{\partial}{\partial x_i^b} \right) R^b \, .$$

For the extended canonical system in Equation (2) we can apply a conservative Lie-series transformation to obtain a new Hamiltonian averaged according to a chosen scheme up to the desired order in the small parameter of the problem (mass of the perturbing body relative to the mass of the central body). With a special choice of generating function S, the resulting formulae do not contain the additional variables and represent in fact the averaged system in Equation (1) in standard mean osculating elements. The averaged equations of motion cannot principally built a canonical system but have a compact quasi-canonical form expressed through the Poisson brackets. For the case of three fully interactive bodies up to second order in the small parameter they can be written in the following form (the upper index (1) marks the new elements): for body a

$$\frac{\partial x_i^{(1),a}}{\partial t} = \frac{\partial < R_0^a + R_1^a + R_2^a >_{\mathrm{s}}}{\partial x_{i+3}^{(1),a}} + \frac{1}{2} < \left[\left\{ \frac{\partial (R_1^a + < R_1^a >_{\mathrm{s}})}{\partial x_{i+3}^{(1),a}}, S_1^b \right\}^b + \left\{ R_1^b + < R_1^b >_{\mathrm{s}}, \frac{\partial S_1^a}{\partial x_{i+3}^{(1),a}} \right\}^b \right] >_{\mathrm{s}},$$

$$\frac{\partial x_{i+3}^{(1),a}}{\partial t} = \frac{\partial < R_0^a + R_1^a + R_2^a >_{\mathrm{s}}}{\partial x_i^{(1),a}} - \frac{1}{2} < \left[\left\{ \frac{\partial (R_1^a + < R_1^a >_{\mathrm{s}})}{\partial x_i^{(1),a}}, S_1^b \right\}^b + \left\{ R_1^b + < R_1^b >_{\mathrm{s}}, \frac{\partial S_1^a}{\partial x_i^{(1),b}} \right\}^b \right] >_{\mathrm{s}}$$

and for body b

$$\frac{\partial x_i^{(1),b}}{\partial t} = \frac{\partial < R_0^b + R_1^b + R_2^b >_{\mathrm{s}}}{\partial x_{i+3}^{(1),b}} + \frac{1}{2} < \left[\left\{ \frac{\partial (R_1^b + < R_1^b >_{\mathrm{s}})}{\partial x_{i+3}^{(1),b}}, S_1^a \right\}^a + \left\{ R_1^a + < R_1^a >_{\mathrm{s}}, \frac{\partial S_1^b}{\partial x_{i+3}^{(1),b}} \right\}^a \right] >_{\mathrm{s}}$$

$$\frac{\partial x_{i+3}^{(1),b}}{\partial t} = \frac{\partial < R_0^b + R_1^b + R_2^b >_{\mathrm{s}}}{\partial x_i^{(1),b}} - \frac{1}{2} < \left[\left\{ \frac{\partial (R_1^b + < R_1^b >_{\mathrm{s}})}{\partial x_i^{(1),b}}, S_1^a \right\}^a + \left\{ R_1^a + < R_1^a >_{\mathrm{s}}, \frac{\partial S_1^b}{\partial x_i^{(1),b}} \right\}^a \right] >_{\mathrm{s}} .$$

Here $< f_i >_{\mathrm{s}}$ stands for the secular part of f which is of order i in the small parameter and the indices at the Poisson brackets denotes in which elements they should be calculated. The terms on the right side are principally new: they visualize in an unexpectedly simple form the fact that the osculating elements do not build a canonical set in the N-body problem. The algorithm can be applied to an arbitrary number of interacting bodies and extended to further approximations. In case of mean-motion resonances the resonant terms should be preserved in the averaged equations of motion for further numerical integration or qualitative dynamical studies.

The simplified form of the algorithm has already been applied to the problem of asteroid motion in the gravitational field of fully interacting perturbing bodies (Tupikova, 2009). It was shown that in the case when only the equations for a massless body have been averaged, they still keep the canonical form at least to order three in the small parameter. Our method revealed also some important new terms that were missed in those theories where not the whole system of differential equations of motion of all the bodies involved has been treated simultaneously in the same algorithm, but an already simplified model for the system of perturbing bodies has been inserted into the equations of asteroid motion.

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