ANALYTICAL MODELING OF THE RIGID INTERNAL MOTIONS OF A THREE-LAYER CELESTIAL BODY THROUGH HAMILTON'S PRINCIPLE

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ABSTRACT. We describe how to construct an analytical approximated representation of the internal rigid motions of a non–rigid three–layer celestial body by using the Hamilton's principle. This method runs parallel to those employed in the Lagrangian or Hamiltonian formulations of Analytical Dynamics. We also discuss the advantages of this approach with respect to other known treatments that tackle this complex problem.

1. INTRODUCTION

The interior of some bodies of the solar system can be approximately reproduced by a non–rigid three–layer model. This model consists on a solid external layer that encloses a fluid containing a solid body (see Figure 1). Indeed, it is the case of the Earth, but other planets or moons might also present the same structure (see, for example, Grinfeld and Wisdom 2005, Hussmann et al. 2006) like Mercury or some icy moons containing a subsurface ocean (Europa, Titania, etc.).

Although all of these bodies share a similar structure, it is necessary to remark that their physical characteristics can be quite different. This is easily understood if we consider, for example, simple models with homogeneous spherical layers for the Earth, Mercury, Titania, and Europa (Grinfeld and Wisdom 2005, Hussmann et al. 2006). The ratio between the density of the fluid and that of the internal solid layer, ρ_F/ρ_S, is close to 1 in the case of the Earth and Mercury, and relatively small for Europa and Titania. This fact is explained taking into account that for the Earth and Mercury the fluid and the internal solid layer have an almost identical iron composition. In contrast, the fluid layer of the models of the icy bodies Europa and Titania is a subsurface ammonia–water ocean, much less dense than the internal solid layer that is composed of silicate rock (Hussmann et al. 2006).

We can also observe other important differences in the relative mass of each layer depending on the particular body. This is shown in Table 1 for the above mentioned bodies, where we have displayed the values for the ratios m_M/m, m_F/m, and m_S/m, the subscripts M, F, and S referring to the external layer, the fluid, and the internal solid layer, respectively. The symbol m denotes the mass of the body and $m_{M,F,S}$ the mass of the respective layer.

Body	m_M/m	m_F/m	m_S/m	ρ_F/ρ_S
Earth	0.63	0.35	0.02	0.92
Mercury	0.31	0.09	0.60	0.84
Titania	0.38	0.04	0.58	0.29
Europa	0.04	0.04	0.92	0.24

Table 1: Some examples of simple models of three–layer bodies of the solar system.

From the point of view of Dynamics, this system presents an interesting feature: even for the simplest models the solid constituents can perform independent rigid internal motions, that is to say, can make independent rotations and translations around the barycenter of the body. Therefore, for these models there can appear differential rotations, or librations, and translations of the internal solid body with respect to the external solid layer.

The interest in investigating those internal motions relies on the fact that their modeling is essential to determine the evolution of the reference systems attached to each celestial body. At the same time, since the rigid motions are affected by the specific characteristics of the body, their observation can constrain, to some extent, the structure of the body (see, for example, Koot 2011), giving valuable information about its interior. Many times it is not possible to obtain this information by other means.

2. COMMON APPROACHES TO STUDY THE DYNAMICS OF A THREE–LAYER BODY

There exist distinct ways to model the dynamical behavior of the rigid motions of a three–layer body (see, for example, Escapa et al. 2001, Escapa and Fukushima 2011, and references therein), whose geometrical and physical configurations are assumed to be quasi–spherical. Some of them are based on the partial differential equations of Continuous Mechanics, such as the elastic–gravitational normal mode theories developed to study the rigid internal motions of the Earth (see, for example, Smith 1977). However, the intrinsic nature of these normal mode methods, which usually are numerical, does not provide much insight into the evolution of the model and its dependence with the physical characteristics of the body. In addition, to profit all the potential of these approaches it is necessary to have detailed density and rheological parameter profiles within the body, what can present some inconveniences from the perspective of Dynamical Astronomy (see, for example, Dehant et al. 1999), or not to be available for some bodies of the solar system.

Other different approaches consider the vectorial form of the evolution of the linear and angular momentum for each layer taken as a whole subsystem. It is the case, for example, of Mathews et al. (1991) when studying the Earth nutations, of Van Hoolst et al. (2008) when treating the Europa librations, or of Grinfeld and Wisdom (2005) when modeling the differential internal translations of a three–layer body, usually referred as the Slichter modes in the Earth terminology. These formulations require the explicit calculation of the forces and torques exerted by each layer on the remaining ones. For instance, it is necessary to compute the hydrodynamical interactions that the fluid exerts on the adjacent solid layers. Since these methods only focus on a part of the internal motions of the system, the rigid part, they are less complete than those based on Continuous Mechanics. Indeed, in their construction it is necessary to make some *a priori* assumptions that approximate to some degree the fluid flow and the elastic deformation field of its constituents. Nevertheless, they offer some advantages due to its relative simplicity, which often makes possible to obtain a clear representation of the influence of the physical characteristics of the body in its motion. This is mainly due to the fact that the rheology of the body is characterized by a small set of parameters that reflects these properties in an averaged sense such as the moments of inertia, Love numbers, etc. It allows the fitting of some of them to the available observations, what is specially meaningful for astronomic and geodetic purposes.

Yet another possible framework to tackle the dynamics of a three–layer body is by means of the variational principles of Mechanics, starting from Hamilton's principle (in its broader sense) and running parallel to the formulations used in Analytical Dynamics. There have been many investigations that have already employed this kind of methods to study the rotational, or librational, motions of non–rigid celestial bodies composed of one or two layers. This is the case of the Earth (see, for example, Poincaré 1910; Jeffreys and Vicente 1957; Moritz 1982; Kubo 1991; Getino 1995; Getino and Ferrándiz 1995, 2001; Ferrándiz et al. 2004; Escapa 2011), but also of other celestial bodies like, for example, Io (Henrard 2008) or Mercury (Noyelles et al. 2010). These variational formulations have been extended to different three–layer bodies, considering the rotational motions (see, for example, Escapa et al. 2000, 2001, 2002) and also the translational ones (Escapa and Fukushima 2011).

The variational methods present the same advantages and limitations as the vectorial treatments when compared with Continuous Mechanics formulations. However, Hamilton's principle theories have also some important gain with respect to the vectorial approaches. Besides the general benefits over the vectorial mechanics methods (see, for example, Lanczos 1986), the variational theories allow to treat the fluid and solid layers as one single dynamical system. It implies that it is not necessary to compute explicitly the hydrodynamical interactions exerted by the fluid on the solid layers (see, for example, Lamb 1963), what simplifies greatly the construction of the equations of motion. In addition the form of these equations is well suited to apply the analytical and numerical mathematical tools developed in other branches of Celestial Mechanics. It makes possible, for example, the construction of consistent higher order analytical approximated solutions or the study of the couplings among the internal and external rigid motions.

3. ANALYTICAL MODELING TROUGH HAMILTON'S PRINCIPLE

Next, we will briefly describe the necessary steps to determine the rigid internal motions of a three–layer celestial body within a variational framework. To get a more complete description on these topics, we refer the reader to the works by Escapa et al. (2001, 2002), Escapa and Fukushima (2011), and references therein.

Hamilton's principle for holonomic dynamical systems (see, for example, Whittaker 1988) states that the value of a certain integral is stationary in the motion. Explicitly, it holds that

$$\int_{t_0}^{t_1} (\delta\mathcal{T} + \delta\mathcal{W})\ dt = 0, \tag{1}$$

where δ denotes the variation, $\mathcal{T}$ is the kinetic energy of the system, and $\mathcal{W}$ is the work done on the system by the external forces that, in the general situation, are non–conservative. From this equation, and following a known process, one obtains a system of differential equations that determines the motion. These differential equations can adopt diverse forms. From the point of view of Celestial Mechanics and focusing on the study of the dynamics of a three–layer celestial body, the most common and useful forms are those derived from a Lagrangian or a Hamiltonian functions of the system.

Let us recall that if the Lagrangian of the system is given by

$$\mathcal{L} = \mathcal{T} - \mathcal{V}, \tag{2}$$

where $\mathcal{V}$ is the potential energy stemming from the conservative forces acting on the system, the equations of motion in terms of a holonomic set of n generalized coordinates q and its associated velocities $\dot{q}$ are

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \left(\frac{\partial\mathcal{L}}{\partial q_i}\right) = \mathcal{Q}_i. \tag{3}$$

Here, $\mathcal{Q}_i$ are the generalized force associated to the coordinate q_i, with $i = 1, ..., n$, included to account for the non–conservative nature of a part of the external interactions.

A related form to these equations is obtained when substituting the generalized velocities $\dot{q}_i$ by n independent linear combination of them ω_i, which define a quasi–coordinates set. This substitution is convenient, for example, in some applications related with the rotation of celestial bodies (see, for example, Moritz 1982, Escapa et al. 2002). By so doing, the equations of motion (see, for example, Whittaker 1988) turn out to be

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\omega_i}\right) + \sum_{j,\,k} c_{ijk}\omega_j \frac{\partial\mathcal{L}}{\partial\omega_k} - \sum_r \beta_{ri}\frac{\partial\mathcal{L}}{\partial q_r} = \mathcal{Q}_i, \tag{4}$$

where c_{ijk} and β_{ri}, with $i, j, r = 1, ..., n$, are functions of q_i, whose explicit expressions can be derived from the relationship between $\dot{q}$ and ω.

Finally, other important way to implement Hamilton's principle is through the Hamiltonian function of the system. In the case of natural systems (see, for example, Whittaker 1988), it takes the form

$$\mathcal{H} = \mathcal{T} + \mathcal{V}. \tag{5}$$

This function depends on n canonical momenta p_i, with $i = 1, ..., n$, and n canonical conjugated coordinates q_i (not being necessarily a holonomic set of generalized coordinates) that describe the dynamical configurations of the system

$$\begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial\mathcal{H}}{\partial q_i} + \mathcal{Q}_{q_i}, \\ \frac{dq_i}{dt} &= \frac{\partial\mathcal{H}}{\partial p_i} - \mathcal{Q}_{p_i}. \end{aligned} \tag{6}$$

The functions $\mathcal{Q}_{p_i}$ and $\mathcal{Q}_{q_i}$ are the canonical generalized forces that must be include in the presence of non–conservative forms. Although sometimes the geometrical or kinematical meaning of a canonical set is not so clear as in the case of the generalized coordinates, the Hamiltonian formalism has been very useful in the field of Celestial Mechanics because of the existence of systematic perturbation methods, such as those based on the Lie series method (Hori 1966), which allows to find an approximation of the solution of the equations of motion.

Regardless the particular formalism that materializes Hamilton's principle, the procedure in which the equations of motion are constructed is quite similar from a formal perspective and can be sequenced as follows:

a) To chose a finite number of generalized coordinates, or canonical variables, that determine the rigid motions of the system. For example, if the solid layers are assumed to be rigid bodies one could take the coordinates of their barycenters to describe their translational motion and some angles, like the Euler ones, to describe their rotational motion. When considering elastic bodies this choice must be adapted by introducing, for example, the concept of Tisserand mean system (see, for example, Escapa 2011). In regard to the fluid, and depending on the particular problem under consideration, the fluid flow can be represented for these purposes by means of a potential motion (see, for example, Escapa and Fukushima 2011), a Poincaré flow (see, for example, Escapa et al. 2001, 2002), etc.

b) To construct the kinetic energy of the system. It is given by the sum of the kinetic energy of its layers

$$\mathcal{T} = \mathcal{T}_M + \mathcal{T}_F + \mathcal{T}_S. \tag{7}$$

In the general situation, and once evaluated the field of the velocities within the body, this computation relies on the modeling of the inertia matrices that depends on the elastic deformation of the layers (see, for example, Getino and Ferrándiz 1995, 2001, Escapa 2011) and also on the deformation of the fluid layer due to the differential motions of the solid constituents (see, for example, Escapa et al. 2001). It is important to underline that in the variational methods the term $\mathcal{T}_F$ is the responsible of modeling the solids–fluid interactions (see, for example, Lamb 1963). So, those interactions are automatically incorporated when constructing the kinetic energy of the system, without the need of computing them separately as it is done in the vectorial treatments.

c) To obtain the potential energy of the system. This function, which has often a gravitational origin, is convenient split into the form

$$\mathcal{V} = \mathcal{V}_{\mathrm{int}} + \mathcal{V}_{\mathrm{ext}}, \tag{8}$$

where $\mathcal{V}_{\mathrm{int}}$ is the internal potential energy of the system, arising from the conservative interactions among the constituents of the non–rigid body. In a similar way, $\mathcal{V}_{\mathrm{ext}}$ is the external potential energy of the system that accounts for the conservative interactions between the body and the external bodies or fields.

d) To compute the generalized, or canonical, forces due to non–conservative forces or torques. These interactions are related with dissipative process usually connected with the rheological properties of the fluid. The construction of these functions requires the evaluation of the virtual work made by the forces, or torques, in a virtual displacement of the system that, depending on the situation, must be expressed in terms of the generalized coordinates, the quasi–coordinates, or the canonical set (see, for example, Getino et al. 2000, Escapa et al. 2002).

e) To form the system of ordinary differential equations that characterize the dynamics of the system. From the expressions of the kinetic energy, the potential energy, and the generalized forces of the system, the computation of the equations of motion is straightforward by applying Equations (3), (4), and (6) depending on the chosen formalism to describe the dynamics of the system.

Once obtained the differential equations of the system, it is necessary to solve them in order to have a quantitative description of the motion. Regrettably, in very few cases it is possible to find an exact analytical solution of this kind of equations. Therefore, bearing in mind the purposes of the investigations one is forced to employ analytical perturbation methods, numerical integration methods, or a combination of both. In this regard, and as it is known from the transformation theory of Dynamics, a convenient choice of the variables used to describe the dynamics of the system can make easier these tasks.

4. DISCUSSION

In the scope of this note, it is not possible to work out in detail the previous procedure for some concrete model. We refer to the reader to the works by Escapa et al. (2001) and Escapa and Fukushima (2011), where he will find two quite different examples of the application of variational methods to the study of the dynamics of three–layer bodies. In the first one, developed within a Hamiltonian formalism, it is investigated the rotational motion of a three–layer Earth model (Figure 1, left side) composed of three nearly spherical, elliptical layers, with common barycenters: an axial–symmetric rigid mantle, an stratified fluid outer core, and an axial–symmetric inner core. For this model the fluid flow is assumed to have uniform vorticity, that is to say, it is considered a Poincaré flow, whereas the solid constituents rotates like rigid bodies. The key point in this case is the construction of the matrix of inertia of the fluid layer, which has a part depending on the rotational variables of the system, since the mantle and the inner core can rotate independently.

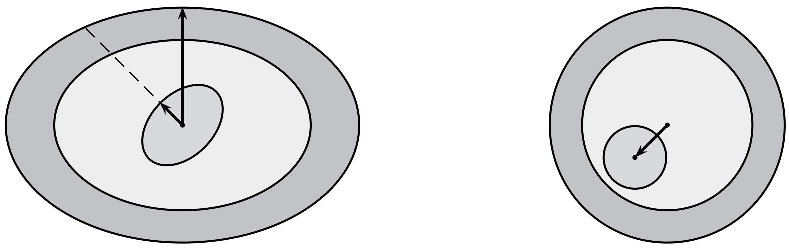

Figure 1: Three–layer models (not scaled) considered in Escapa et al. (2001) and Escapa and Fukushima (2011).

In the second one, constructed within a Lagrangian formalism, it is worked out the internal translational motion of a body differentiated into three homogeneous layers (Figure 1, right side) with spherical symmetry: an external ice-I layer, a subsurface ammonia–water ocean, and a rocky inner core. This is the basic structure of three–layer icy bodies. In contrast to the rotational situation, here the fluid motion is entirely due to the translational motion of the solid rigid layers. It implies that the fluid flow is irrotational and can be derived from a velocity potential. This velocity potential is a solution of the Laplace equation in the fluid domain with the proper boundary conditions.

Both examples lead to analytical solutions, since the equations of motions are linearized around a periodic motion or an equilibrium position. In this way, the rigid motions are characterized by a set of proper frequencies (rotational and translational normal modes) that explicitly depends on some parameter describing the properties of the models like its moments of inertia, masses, and densities. This allows to perform a detailed study of the dependence of those frequencies on the physical characteristics of the model and *vice versa*.

In conclusion, Hamilton's principle formalisms constitute a convenient approach to model the rigid internal motions of a three–layer celestial body. In the variational methods, the dynamics is constructed from the kinetic energy, the potential energy, and the generalized forces of the system, the physical specification of the body being determined by a small set of parameters. One of the main advantages of these treatments relies on the consideration of the solid and fluid layers as forming one single dynamical system. It implies that it is not necessary to compute explicitly the fluid-solids hydrodynamical interactions.

In addition, the form of the differential equations of the motion allows to apply the mathematical tools of Celestial Mechanics, what is specially relevant when one aims at obtaining an analytical approximated description. Moreover, the Hamiltonian formalism is particularly appropriated to construct higher order perturbation solutions in a systematic and consistent way. These kind of solutions can be helpful, for example, to establish standard models, like in the Earth rotation theory case, or to check the numerical codes employed by other treatments.

At any rate, it is important to remark that the intrinsic complexity of the motions of a non–rigid three–layer celestial body makes advisable, if not necessary, to develop different and independent approaches in order to compare the results derived by each of them. It will improve our understanding on the dynamics of these systems.

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