ANALYTICAL COMPUTATION OF THE EFFECTS OF THE CORE-MANTLE BOUNDARY TOPOGRAPHY ON TIDAL LENGTH-OF-DAY VARIATIONS

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ABSTRACT. We have computed coupling mechanisms at the core-mantle boundaries of terrestrial bodies of the Solar system, and in particular, the pressure torque on the topography at the core-mantle boundary. The philosophy of the computation follows Wu and Wahr (1997), which allows to solve for the velocity field coefficients in terms of the topography coefficients. The velocity in the fluid core is decomposed into a global classical velocity and an incremental velocity related to the topography. We have used an analytical approach to compute this last part as well as the incremental changes in the periodic variations of the length-of-day (LOD) and in the librations, i.e. oscillating motions in space. We have found that there are topography coefficients that are enhanced due to resonances at particular frequencies. For the Earth the tidal forcing frequencies are compared to those resonance frequencies. The total torque on the core-mantle boundary is demonstrated to be dependent on particular amplitudes of the topography.

1. MOTIVATION

The length-of-day variations are usually computed from angular momentum equations of the whole Earth and of the different layers (inner core, outer core and mantle). Coupling mechanisms must be considered at the CMB and ICB (Inner Core Boundary). The torque considered in the classical approach is the gravitational and pressure torques related to the flattening of the core. In a more sophisticated approach and for the nutations, one considered the electromagnetic torque (e.g. Mathews et al., 1991) or even the viscous torque (e.g. Mathews and Guo, 2005). The topographic torque related to the non-hydrostatic part of the CMB shape is not considered, while the problem has been addressed by Wu and Wahr (1997) in a numerical approach. These

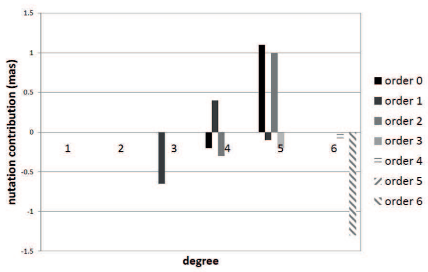

Figure 1: Results concerning nutations, from Wu and Wahr, 1997

authors have shown that the topographic torque, often disregarded, may be important at CMB. Wu and Wahr (1997) have computed numerically the topographic torque using topography expanded in Spherical Harmonics and have shown that some harmonics of the topography have important contributions to nutations. From the approach and from the results, it is difficult to understand if the enhancement of some topography components is due to the topography amplitudes or due to something else, such as the geometry of the core topography itself. We may suspect some resonance effects with inertial waves as when perturbing a rotating fluid, the particle motion is characterized by a low-frequency oscillation called inertial wave. We may thus wander if the enhancements seen in the topographic torque would be related to enhancements of the inertial waves. For that reason we have considered to use an analytical approach in order to study the results in more detail. In this paper we consider the approach in the frame of length-of-day variations. Nutations are computed elsewhere (see Dehant and Folgueira, 2011).

2. EQUATIONS

The total pressure torque on the whole topography can be decomposed into two parts: $\Gamma^0 + \Gamma^{\phi}_{topo}$, where (1) Γ^0 the constant classical part of the torque for an ellipsoidal topography at equilibrium, and (2) Γ^{ϕ}_{topo} due to the inertial rotation pressure $(P - P_0 - \rho_f \phi^f)$ related for the global relative rotation of the fluid on the topography, where P is the pressure and $\phi^f = \phi^e + \phi^1$ is the gravitational potential consisting of the sum of the external potential (ϕ^e) and the incremental potential due to the deformation (ϕ^1). Only the second part of the torque is of importance when computing the effects of a perturbing potential and related additional rotations of the core and the mantle on a topography different with respect to the ellipsoidal hydrostatic shape.

In order to be able to evaluate this torque, one considers the expression $\vec{\Gamma}^{\phi}_{topo} = -\int\int_{CMB} \vec{r} \times \vec{n} \rho_f \, \phi \, dS$ related to $\phi \left(= \frac{(P-P_0)}{\rho_f} - \phi^f\right)$. In order to integrate the torque, one needs then to consider the Navier-Stokes equation. The philosophy for solving the equations is to separate the velocity into a global part ($\vec{v}$) and an additional part ($\vec{u}$ or $\vec{q}$ if normalized) and to separate the equation into two equations of which the solutions are $\vec{v}$ and $\vec{q}$ and can be computed analytically. $\vec{v}$ is the global relative velocity of the fluid and $\vec{q}$ is the incremental one, function of the coefficients a_l^k in the expression of ϕ, both being incompressible.

The boundary conditions at the CMB are imposed on the total velocity and yield thus a relation between $\vec{v}$ (and thus components of the relative global fluid rotation m_1^f and m_2^f), $\vec{q}$ (and thus the a_l^k coefficients), and the topography coefficients ϵ_n^m. This allows to solve for the a_l^k in terms of the relative global relative fluid rotation.

The basic dynamical equations are the linearized Navier-Stokes equation. If one considered that the equilibrium corresponds to the hydrostatic case, they become:

$$\frac{\partial \vec{V}}{\partial t} + 2\vec{\Omega} \times \vec{V} + \frac{1}{\rho_f} \nabla p - \nabla \phi_m + \Omega \frac{\partial \vec{m}}{\partial t} \times \vec{r} = 0 \tag{1}$$

where $\vec{\Omega}$ is the uniform equilibrium angular rotation of amplitude Ω, $\vec{m}$ is the scaled additional mantle angular velocity, $\vec{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$, $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the position of the fluid particle in the reference frame, $\vec{V}$ is the velocity of the fluid particle in the reference frame, ρ_f is the fluid density and p is the incremental effective pressure computed by $p = P - P_0 - \rho_f \phi_1 - \rho_f \phi_e$ as expressed before. Note that the angular velocity vector of the reference frame attached to the mantle $\vec{\omega} = \vec{\Omega} + \Omega \vec{m}$.

The first boundary condition at the core-mantle boundary (CMB): $\vec{n} \cdot \vec{V} = 0$ ($\vec{n}$ is the normal to the surface); it is expressed as a function of the boundary topography; the boundary surface (hydrostatic + non-hydrostatic parts) is expressed using:

$$r = r_0 \left[1 + \sum_{n=1} \sum_{m=-n}^{n} \varepsilon_n^m Y_n^m(\theta, \lambda) \right] \tag{2}$$

where r_0 is the surface mean radius, $Y_n^m(\theta, \lambda)$ are the Legendre Associated Functions as a function of the colatitude θ and the longitude λ, and ε_n^m are small dimensionless numbers related to the existence of the topography. The largest contribution is ε_2^0 due to the flattening (hydrostatic + non-hydrostatic parts) of the CMB. It must be noted that the ϵ_2^0 in a topography development in spherical harmonics usually contains the hydrostatic part and the non-hydrostatic contribution to the topography; these must be separated. Here it is separated into a hydrostatic part $\epsilon_2^{0\;hydr}$ and an additional one noted confusingly ϵ_2^0 as well from now on and for writing simplicity.

The second boundary condition is the condition of incompressibility: $\nabla \cdot \vec{V} = 0$. We now decompose the velocity: $\vec{V} = \vec{u} + \vec{v} = \Omega\, L\, \vec{q} + \vec{v}$, where L is the maximum radius of the core and $\vec{q}$ is a non-dimensional velocity. One imposes that $\vec{u} << \vec{v}$.
The equation and condition for $\vec{v}$ are:

$$\begin{cases} \frac{\partial \vec{v}}{\partial t} + 2\vec{\Omega} \times \vec{v} + \; + \Omega\, \frac{\partial \vec{m}}{\partial t} \times \vec{r} - \nabla \phi_m \; = \; 0 \\ \nabla \cdot \vec{v} = 0 \end{cases} \tag{3}$$

The equation and condition for $\vec{q}$ are:

$$\begin{cases} i\,\sigma_m\, \vec{q} + 2\vec{\hat{z}} \times \vec{q} + \nabla \Phi \; = \; 0 \\ \nabla \cdot \vec{q} \; = \; 0 \\ \vec{n} \cdot \vec{q} + \Omega^{-1}\, L^{-1} \vec{n} \cdot \vec{v} \; = \; 0 \end{cases} \tag{4}$$

where $\Phi = \frac{\phi}{\Omega^2\, L^2}$ and $\phi = \frac{p}{\rho_f}$, Φ being called the non-dimensional dynamic pressure. The time dependence of the variables is considered as $e^{i\sigma t}$ where σ is the nutation frequency in the reference frame attached to the mantle. When used in non-dimensional equations as above, the frequency to be used is σ_m instead of σ, where $\sigma = \Omega \sigma_m$. In the case of computing the Length-of-day variations, the velocity $\vec{v}$ can be expressed by$\vec{v} = -\Omega m_3 \vec{\hat{z}} \times \vec{r}$.

After some algebra on the first line of Equation (4), one can obtain the following expression for $\vec{q}$ as a function of $\nabla\Phi$:

$$\vec{q} \;=\; \frac{-i\,\sigma_m}{4 - \sigma_m^2} \left[\nabla \Phi - \frac{2}{i\,\sigma_m} \vec{\hat{z}} \times \nabla \Phi - \frac{4}{\sigma_m^2} (\vec{\hat{z}} \cdot \nabla \Phi)\, \vec{\hat{z}} \right] \tag{5}$$

where $\vec{\hat{z}}$ is the normalized vector in the direction of $\vec{\Omega}$.

Using the above equation for $\vec{q}$ and the incompressibility condition for this fluid velocity (second line of Equation 4), one obtains the following equation for Φ: $\nabla^2 \Phi - \frac{4}{\sigma_m^2} \frac{\partial^2 \Phi}{\partial Z^2} \; = \; 0$ where Z is a particular coordinate (related to the cylindrical coordinates involving the colatitude θ), which is equal to $\sqrt{\frac{\sigma_m}{2}} \cos\theta$. The factor $(1 - \frac{4}{\sigma_m^2})$ being negative, this mixed differential equation is an hyperbolic differential equation has the typical form of a wave propagation equations. It expresses that small perturbations of an equilibrium configuration can propagate in the fluid in the form of waves which are the so-called inertial waves because they are controlled by the Coriolis force as a restoring force.

The solution of this equation for Φ must be proportional to the associated Legendre functions of the first kind; it has the following form:

$$\Phi \;=\; \sum_{l=1} a_l^k\, P_{lk}(\frac{\sigma_m}{2})\, Y_l^k(\theta, \lambda). \tag{6}$$

where $P_{lk}(\frac{\sigma_m}{2})$ are the fully normalized associated Legendre polynomials, and the $Y_l^k(\theta, \lambda)$, the fully normalized associated Legendre functions as introduced before, and the a_l^k are coefficients that will be determined in the next step using the boundary conditions (third line of Equation 4). Using the boundary condition for $\vec{q}$ (third line of Equation 4) and the expression of $\vec{q}$ in function of Φ (Equation 5), substituting the above solution for Φ (Equation 6), after a lot of algebra, one obtains for the first order in the small quantities such as ϵ_n^m:

$$\sum_{l,k} Y_l^k \left[k P_{lk}(\frac{\sigma_m}{2}) - \left(1 - \frac{\sigma_m^2}{4}\right) P'_{lk}(\frac{\sigma_m}{2}) \right] a_l^k - 2\left(1 - \frac{\sigma_m^2}{4}\right) \sum_{n=1} m \epsilon_n^m Y_n^m m_3 = 0 \tag{7}$$

where $Y_l^k \equiv Y_l^k(\theta, \lambda)$. Equation (7) allows us to solve for the a_l^k as a function of the ϵ_n^m and σ_m; because we have only kept first order in ϵ_n^m, the a_l^k coefficients are linear functions of ϵ_n^m. It must be noted that this equation can be considered component per component by projection on each $Y_{l'}^{m'}$ and that we can solve as well for each ϵ_n^m separately and then sum over all the contributions.

3. RESULTS

Substituting the solution for Φ, provided in Equation (6) as a function of the coefficients a_l^k, in the expression for $\vec{q}$ provided by Equation (5), and computing the contribution to the torque, one gets the $\vec{q}$-contribution to topographic torque $\vec{\Gamma}_{topo}^{\phi}$ as a function of a_l^k (or equivalently ϵ_n^m by means of Equation 7). The results are shown in Figure 2.

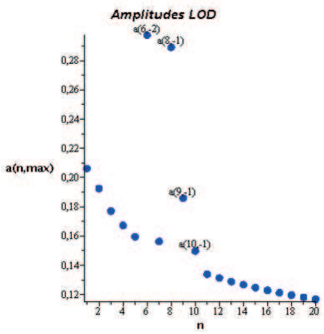

Figure 2: Amplitude of the a_l^k for a_l^k being the maximum values for each n.

4. CONCLUSIONS

From our computation we have seen that some topography coefficients provide larger contributions to LOD than others and that there are some differences with respect to Wu and Wahr (1997), even when using the same CMB topography. But the main features such as the degree 6 being larger than the others, has been recovered.

With this computation, we have understood that the degrees and orders of these amplifications do not depend on the geometry/dimension/topography amplitudes of core but rather on the degrees and orders of the excitation and topography expressed in spherical harmonics.

We must note however that our computations/conclusions may change with an inner core.

5. REFERENCES

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