ASYMMETRIC EXCITATION OF THE POLAR MOTION

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ABSTRACT. Polar motion analysis is commonly based upon symmetric linearised Euler-Liouville equations. Then, in absence of forcing, the rotation pole coordinates evolve in the same way. Actually, expressing thoroughly the pole tide and taking into account the triaxiality, the equations become asymmetric with respect to the pole coordinates. This leads to the formulation of the generalised Euler-Liouville equation, for which we derive a general solution. We discuss possible observational consequences.

1. INTRODUCTION

The geophysical analysis of the polar motion is generally accomplished thanks to the *symmetric* linearised Euler-Liouville equations in the Terrestrial Reference Frame (TRF):

$$m + \frac{i}{\tilde{\sigma}_c}\dot{m} = \Psi \tag{1}$$

where $m = m_1 + im_2$ is the complex equatorial coordinate of the instantaneous rotation pole and $\tilde{\sigma}_c = \sigma_c(1 + \frac{i}{2Q})$ the complex Chandler angular frequency of which the quality factor Q (within the range $40 - 200$) accounts for dissipation. There Ψ means the modelled equatorial excitation, produced by mass transports within the Earth or its hydro-atmospheric layers. In this equation m_1 and m_2 are driven in the same way by geophysical excitation. Actually it neglects any asymmetric effects resulting from triaxiality and rotational deformation. Whereas triaxiality has been investigated by many studies (see e.g. Chen and Shen 2010), the asymmetry brought by ocean pole tide is oddly overlooked. The consistent analysis of both effects is done. This leads to an extended form of equatorial Euler-Liouville equation, for which we propose a general solution.

2. ASYMMETRIC EFFECTS

We start from the Euler-Liouville equation for a triaxial Earth expressed in the frame of the mean principal axes $Gx'y'z'$ associated with inertia moment $A < B < C$ (Munk and Mac Donald 1960):

$$m'_1 - \frac{B}{(C-A)\Omega}\dot{m}'_2 = \Psi'_1 \qquad m'_2 + \frac{A}{(C-B)\Omega}\dot{m}'_1 = \Psi'_2 \tag{2}$$

where m'_1 and m'_2 are rotation pole coordinates related to $Gx'y'z'$, and Ψ'_1 and Ψ'_2 are the components of the equatorial excitation function in $Gx'y'z'$. We adopt the values $A = 8.010083(9)\,10^{37}$ kg m^2, $B = 8.010260(9)\,10^{37}$ kg m^2, $C = 8.036481(9)\,10^{37}$ kg m^2 derived by Chen and Shen (2010) from EGM08 gravity model. Note that $(B-A)/A \approx 2,5\,10^{-5}$ whereas $(A-C)/A \approx (B-C)/B \approx 3\,10^{-3}$. The equatorial excitation function is given by:

$$\Psi'_1 = \frac{\Omega c'_{13} + h_1}{\Omega(C-A)} + \frac{\Omega \dot{c}'_{23} + \dot{h}'_2}{\Omega^2(C-A)} - \frac{L'_2}{\Omega^2(C-A)}\ ; \quad \Psi'_2 = \frac{\Omega c'_{23} + h_2}{\Omega(C-B)} - \frac{\Omega \dot{c}'_{13} + \dot{h}'_1}{\Omega^2(C-B)} + \frac{L'_1}{\Omega^2(C-B)} \tag{3}$$

where $c_{1/2,3}$ are the off-diagonal inertia moment increments, $h_{1/2}$ the components of the relative angular momentum and $L_{1/2}$ the external torque. The triaxiality affects the geophysical function at the level of 1%. Insofar as the modelling of the latter effect has a much larger relative uncertainty, the difference between $C - A$ and $C - B$ can be cast aside. Let $\bar{A}$ be the mean equatorial moment given by:

$$\bar{A} = (A+B)/2 \tag{4}$$

then, the equatorial geophysical function is approximated by:

$$\Psi' = \Psi'_1 + i\Psi'_2 = \frac{\Omega c' + h'}{\Omega(C-\bar{A})} - \frac{i}{\Omega}\frac{\Omega \dot{c}' + \dot{h}'}{(C-\bar{A})\Omega} + i\frac{L}{\Omega^2(C-\bar{A})} \tag{5}$$

After having introduced the *triaxial coefficients*:

$$r = \sqrt{\frac{(C-A)A}{(C-B)B}} = 1.00379 \ , \ \ \Delta r = r - 1 \approx 3.8\,10^{-3} \tag{6}$$

the equations (2) can be shortened into the form:

$$m_1' - \frac{1-\Delta r}{\sigma_e}\dot{m}_2' = \Psi_1' \qquad\qquad m_2' + \frac{1+\Delta r}{\sigma_e}\dot{m}_1' = \Psi_2' \tag{7}$$

with $\sigma_e = \sqrt{\frac{(C-A)(C-B)}{AB}}\Omega = e\ \Omega$ (second order terms in Δr are neglected). For practical purpose we have to go back to the Terrestrial Reference Frame $Gxyz$. In a first approximation the triaxial frame can be deduced from the TRF by the axial rotation of angle $\lambda_A = -14.92851(8) \pm 0.0010°$ (Chen and Shen 2010). So for going back to TRF, we apply the complex coordinate change $m = m' e^{i\lambda_A}$ (correspond to the axial rotation of angle $-\lambda_A$ which brings inertia axis Gx' in coincidence with Gx). In the TRF we have also $c = c' e^{i\lambda_A}$. Finally we obtain:

$$m_1 - \frac{1-\Delta r\cos 2\lambda_A}{\sigma_e}\dot{m}_2 - \frac{\Delta r \sin 2\lambda_A}{\sigma_e}\dot{m}_1 = \Psi_1; \quad m_2 + \frac{1+\Delta r\cos 2\lambda_A}{\sigma_e}\dot{m}_1 + \frac{\Delta r\sin 2\lambda_A}{\sigma_e}\dot{m}_2 = \Psi_2 \tag{8}$$

with λ_A the longitude of the first principal inertia axis and $\Psi = \Psi_1 + i\Psi_2$ the geophysical excitation expressed in TRF:

$$\Psi = \Psi_1 + i\Psi_2 = \frac{\Omega c + h}{\Omega(C-\bar{A})} - \frac{i}{\Omega}\frac{\Omega\dot{c} + \dot{h}}{(C-\bar{A})\Omega} + i\frac{L}{\Omega^2(C-\bar{A})} \tag{9}$$

In a first approach we neglect the influence of the fluid core. Let $\tilde{k}_2 = k_2 + i\Bbbk_2 \approx 0.3(1+i0.01)$ be the solid Earth Love number, the rotational excitation associated with the solid Earth is expressed by:

$$\chi^r = \frac{\tilde{k}_2}{k_s} m \tag{10a}$$

By analogy the rotational angular momentum function caused by ocean pole tide is

$$\chi_o^r = \frac{\tilde{k}_o}{k_s}\left[A_1 m_1 + A_2 m_2 + i(A_2 m_1 + B_2 m_2)\right] \tag{10b}$$

where we have introduced the equivalent oceanic Love number $\tilde{k}_o = k_o + i\Bbbk_o$ with the real part

$$k_o = \frac{3}{5}(1 + k_2 - h_2)\frac{\rho_o}{\rho_\oplus}(1 + k_2') \approx 0.05 \tag{11}$$

There $k_2 = 0.3$, $h_2 = 0.6$ and $k_2' = -0.3$, $\rho_o \sim 1035$ kg/m^3 is the ocean density, and $\rho_\oplus = 5500$ kg/m^3 the Earth density. According to this notation, the Desai's model of equilibrium ocean pole tide (Desai 2002, Eq. 24) is associated with the coefficients $A_1 = 0.942 \quad B_1 = -0.021 \quad B_2 = 0.746$.

3. GENERALIZED EULER-LIOUVILLE EQUATIONS

Removing the rotational excitations (10a) and (10b) from the right hand side of (8) and putting them into the left hand side, the equatorial Euler-Liouville equations take the generalised form:

$$\begin{aligned} (1+\alpha_1)m_1 - \tfrac{1+\beta_1}{\sigma_e}\dot{m}_2 + \gamma_1 m_2 + \tfrac{\delta_1}{\sigma_e}\dot{m}_1 &= \Psi_1^{(pure)} \\ (1+\alpha_2)m_2 + \tfrac{1+\beta_2}{\sigma_e}\dot{m}_1 + \gamma_2 m_1 + \tfrac{\delta_2}{\sigma_e}\dot{m}_2 &= \Psi_2^{(pure)} \end{aligned} \tag{12a}$$

with the particular coefficients

$$\begin{array}{lll} e' = \frac{\sigma_e}{\Omega} & \alpha_1 = -\frac{k_2 + k_o A_1 - \Bbbk_o B_1}{k_s} & \alpha_2 = -\frac{k_2 + k_o B_2 + \Bbbk_o B_1}{k_s} \\ & \beta_1 = -\Delta r\cos 2\lambda_A + e'\frac{k_2 + k_o B_2 + \Bbbk_o B_1}{k_s} & \beta_2 = \Delta r\cos 2\lambda_A + e'\frac{k_2 + k_o A_1 - \Bbbk_o B_1}{k_s} \\ & \gamma_1 = \frac{\Bbbk_2 - k_o B_1 + \Bbbk_o B_2}{k_s} & \gamma_2 = -\frac{\Bbbk_2 + k_o B_1 + \Bbbk_o A_1}{k_s} \\ & \delta_1 = -(\Delta r\sin 2\lambda_A + e'\frac{\Bbbk_2 + k_o B_1 + \Bbbk_o A_1}{k_s}) & \delta_2 = \Delta r \sin 2\lambda_A + e'\frac{-\Bbbk_2 + k_o B_1 - \Bbbk_o B_2}{k_s} \end{array} \tag{12b}$$

By contrast to (1) these equations exhibit an asymmetry with respect to m_1 and m_2 and cannot be reduced to a complex form. The generic form (12a) define the *Generalised Linearised Euler-Liouville*

Equations. In the considered case here-above, the coefficients of these equations respect the following orders of magnitude $|\alpha_i| \lesssim 0.3 \quad |\beta_i|, |\gamma_i|, |\delta_i| \lesssim e$ (e= flattening).

Solution in frequency domain (12a) gives two eigenfrequencies. The positive one is associated with the Chandler angular frequency including the damping:

$$\tilde{\sigma}_c \approx \sigma_e \left[\sqrt{(1+\alpha_1)(1+\alpha_2)} + i\frac{1}{2}\left(\gamma_2 - \gamma_1 + \delta_1(1+\alpha_2) + \delta_2(1+\alpha_1)\right) \right] \tag{13}$$

The second eigenfrequency is the complex conjugate of $\tilde{\sigma}_c$:

$$\tilde{\sigma}_c^- = -\sigma_c^* \tag{14}$$

In frequency domain the solution is given by:

$$\left| \begin{array}{l} m_1(\sigma) \\ m_2(\sigma) \end{array} \right. \approx -\frac{\sigma_e^2}{(\sigma-\tilde{\sigma}_c)(\sigma-\tilde{\sigma}_c^-)} \left| \begin{array}{l} (1+\alpha_2+i\frac{\sigma}{\sigma_e}\delta_2)\Psi_1(\sigma) + \left(i\frac{\sigma}{\sigma_e}(1+\beta_1)-\gamma_1\right)\Psi_2(\sigma) \\ (1+\alpha_1+i\frac{\sigma}{\sigma_e}\delta_1)\Psi_2(\sigma) - \left(i\frac{\sigma}{\sigma_e}(1+\beta_2)+\gamma_2\right)\Psi_1(\sigma) \end{array} \right. \tag{15}$$

Let $\Psi(t) = \Psi_0 e^{i\sigma_0 t}$ be a circular excitation at angular frequency σ_0. From (15) its effect on on polar motion is:

$$\begin{aligned} m(t) = &-\frac{\Psi_0\sigma_e^2}{(\sigma_0-\tilde{\sigma}_c)(\sigma_0-\tilde{\sigma}_c^-)}\left[\left(2+\alpha_1+\alpha_2+i(\gamma_1-\gamma_2)+\frac{\sigma_0}{\sigma_e}(2+\beta_1+\beta_2)+i\frac{\sigma_0}{\sigma_e}(\delta_1+\delta_2)\right)\frac{e^{i\sigma_0 t}}{2}\right. \\ &\left.+\left(\alpha_2-\alpha_1-i(\gamma_1+\gamma_2)-\frac{\sigma_0}{\sigma_e}(\beta_2-\beta_1)-i\frac{\sigma_0}{\sigma_e}(\delta_2-\delta_1)\right)\frac{e^{-i\sigma_0 t}}{2}\right] \end{aligned} \tag{16a}$$

where we identify the quantities A^+, A^-, m_0^+ and m_0^- by:

$$m(t) = -\frac{\Psi_0\sigma_e^2}{(\sigma_0-\tilde{\sigma}_c)(\sigma_0-\tilde{\sigma}_c^-)}\left(A^+e^{i\sigma_0 t}+A^-e^{-i\sigma_0 t}\right) = m_0^+e^{i\sigma_0 t}+m_0^-e^{-i\sigma_0 t} \tag{16b}$$

4. POSSIBLE OBSERVATIONAL CONSEQUENCE

We apply the previous formalism to the case of a triaxial Earth partially covered by the oceans. Putting the corresponding coefficients (12b) into (13), the Chandler angular frequency is:

$$\tilde{\sigma}_c \approx \sigma_e\left(1-\frac{\tilde{k}_2}{k_s}-\frac{\tilde{k}_o}{k_s}\frac{A_1+B_2}{2}\right) \tag{17}$$

After having introduced the *effective Love number* $\tilde{k} = \tilde{k}_2 + \tilde{k}_o\frac{A_1+B_2}{2}$ the Chandler frequency takes the classic form $\tilde{\sigma}_c \approx \sigma_e\left(1-\frac{\tilde{k}}{k_s}\right)$. According to (16) a circular excitation at frequency σ_0 produces two circular components in polar motion with opposite frequencies, that is an elliptical motion. The term circling in the same direction as the excitation has the complex amplitude:

$$m_0^+ = -\frac{\Psi_0\sigma_e^2}{(\sigma_0-\tilde{\sigma}_c)(\sigma_0-\tilde{\sigma}_c^-)}A^+ \tag{18}$$

with

$$A^+ \approx \left(1-\frac{\tilde{k}_2}{k_s}-\frac{\tilde{k}_o}{k_s}\frac{A_1+B_2}{2}\right)^* + \frac{\sigma_0}{\sigma_e} = \frac{\sigma_c^*+\sigma_0}{\sigma_e} = \frac{\sigma_0-\tilde{\sigma}_c^-}{\sigma_e} \tag{19}$$

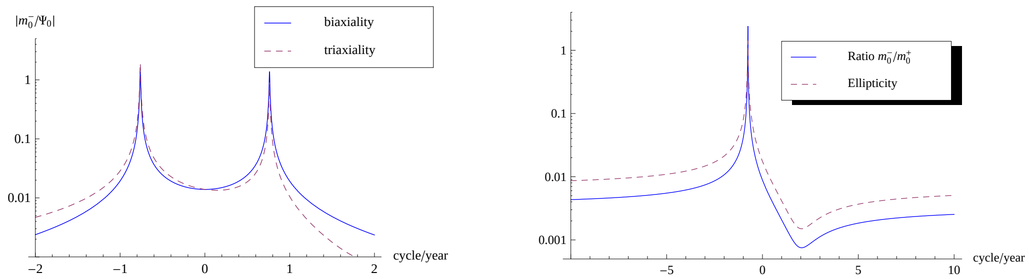
Thus:

$$m_0^+ = -\frac{\Psi_0\sigma_e}{(\sigma_0-\tilde{\sigma}_c)} \tag{20}$$

and we recognise the classic term (or symmetric), exhibiting the unique resonance at Chandler angular frequency $\tilde{\sigma}_c$. But there appears also an exotic term of opposite frequency $-\sigma_0$ given by the complex amplitude:

$$m_0^- = -\frac{\Psi_0\sigma_e^2}{(\sigma_0-\tilde{\sigma}_c)(\sigma_0-\tilde{\sigma}_c^-)}A^- \tag{21}$$

with
$$A^- \approx \frac{k_o}{k_s}\frac{A_1 - B_2}{2} + i\frac{k_o}{k_s}B_1 - \frac{\sigma_0}{\sigma_e}\left(\Delta r e^{i\lambda_A} + e'\frac{k_o}{k_s}\frac{A_1 - B_2}{2} + ie'\frac{k_o}{k_s}B_1\right) \tag{22}$$
Although the term m_0^- presents a double resonance, both at Chandler frequency and its opposite, it is strongly reduced by the smallness of the coefficient A^- (same order than e). We compute the ratio m_0^-/Ψ_0 in two cases: i) oceans and triaxiality are considered together ii) triaxiality is neglected. Results, displayed in Figure 1a, show that m_0^-/Ψ_0 reaches about 3 mas at the resonance frequencies, and mostly results from the oceans alone (biaxiality). The relative impact of m_0^- with respect to the classical effect m_0^+ is quantified by the ratio m_0^-/m_0^+, represented in Figure 1b, and completed by the corresponding ellipticity of the induced polar motion, given by the relative difference between small and great axes:
$$\frac{|m_0^+| + |m_0^-| - (|m_0^+| - |m_0^-|)}{|m_0^+| + |m_0^-|} = \frac{2|m_0^-|}{|m_0^+| + |m_0^-|} \tag{23}$$
The ratio m_0^-/m_0^+ reaches a maximum of 3 at $-\sigma_c$ (1 for ellipticity). Far from this frequency it remains less than 0.05 (0.01 for ellipticity). Considering the retrograde annual term of the polar motion (10 mas),



(a) (b)

Figure 1: (a) Complex ratio m_0^-/Ψ_0 (amplitude) in function of the excitation frequency, exhibiting the double resonance at Chandler frequency and its opposite. (b) Amplitude of the ratio m_0^-/m_0^+ and corresponding ellipticity as function of the frequency.

we see that the asymmetric effect can reach 0.5 mas. On the other hand the geodetic excitation function is radically modified in the vicinity of the Chandler frequency, as we havw shown in Bizouard (2012).

5. CONCLUSION

Pole tide excitation and Earth triaxiality introduce asymmetry which cannot be neglected in light of the contemporaneous pole coordinates accuracy (0.1 mas). Their consistent handling leads to an extended form of the linearised Euler-Liouville equation, for which we propose a general solution. Casting aside the influence of the fluid core, we analyse possible observational consequence of the asymmetric effect. A given circular excitation gives an elliptical polar motion, the ellipticity reaching 1 in the vicinity of the negative Chandler frequency. Quantification of these effects strongly rely on ocean pole tide modelling. A complete derivation and the consequence on geodetic excitation can be found in Bizouard (2012).

6. REFERENCES

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