Application of the spectral analysis methods for the investigation of the Moon rotation

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The aim of the research:

1) **Choice and comparison of the optimal spectral analysis schemes** for the investigation of the rotational motion of the Moon.

2) **Investigation of the discrepancies** between the high-precision numerical solutions of the Moon rotation problem and the semi-analytical theory of the Moon's rotation (SMR), which consists of Cassini relations and the semi-analytical solutions of the lunar physical libration's problem (Eckhardt D.H., 1981), (Moons M., 1982), (Moons M., 1984), (Pešek I., 1982), by the spectral analysis methods.

3) **Construction of the new high-precision Moon Rotation Series (MRS2010)** dynamically adequate to the DE200/LE200 ephemeris over 400 years.

This investigation is carried out only for the Newtonian case.
ALGORITHM:

1. **Numerical solution of the Moon rotation** is implemented with the quadruple precision of the calculations. The initial conditions are computed by the semi-analytical theory of the Moon rotation (SMR). Discrepancies between the numerical solution and SMR are obtained in Euler angles over 400 year time interval with one day spacing.

2. **Investigation of the discrepancies** is carried out by the least squares method (LSQ) and by the spectral analysis (SA) method. The set of the frequencies of SMR theory is used without change in this investigation. Only the coefficients of the periodic terms are improved. The periodic terms representing the new high-precision Moon rotation series **MRS2010** are determined.

3. **Numerical solution of the Moon rotation** is constructed anew with the new initial conditions, which are calculated by MRS2010. Discrepancies of the comparison of the numerical solutions with MRS2010 do not surpass 20 mas over 400 year time interval, that means a good consistency of **MRS2010** series with the DE200/LE200 ephemeris.
Lagrange differential equations of the second kind of the Moon rotation with respect to the fixed ecliptic and equinox of epoch J2000:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\lambda}_i} - \frac{\partial L}{\partial \lambda_i} = 0, \quad i = 0, 1, 2, 3.
\]

where \( L = T + U \), \( T = \frac{1}{2} (A\omega_1^2 + B\omega_2^2 + C\omega_3^2) \),

\( U \) — the force function of the gravitational interaction of the Moon with the disturbing bodies (the Earth, Sun and large planets).

Problem expressed in the Rodrigues – Hamilton parameters:

\[
\lambda_0 = \cos \frac{\theta}{2} \cos \frac{\psi + \phi}{2}, \quad \lambda_1 = \sin \frac{\theta}{2} \cos \frac{\psi - \phi}{2}, \quad \lambda_2 = \sin \frac{\theta}{2} \sin \frac{\psi - \phi}{2}, \quad \lambda_3 = \cos \frac{\theta}{2} \sin \frac{\psi + \phi}{2},
\]

which are functions of the Euler angles \( \psi, \theta \) and \( \phi \).
The force function $U$ of the gravitational interaction of the Moon with the disturbing bodies (the Earth, Sun and large planets) is limited here to the spherical harmonics with coefficients $C^*_{j0}$ for $j=2,\ldots,4$, $C^*_{22}$, $C^*_{3k}$, $S^*_{3k}$ for $k=1,2,3$ and $C^*_{4k}$, $S^*_{4k}$ for $k=1,\ldots,4$. All these coefficients refer to the principal inertia axes of the Moon and are calculated from the non-normalized coefficients $C_{jk}$, $S_{jk}$ of the selenopotential model by Eckhardt D.H., 1981.

The orbital motions of the disturbing bodies are defined by the DE200/LE200 ephemeris.

The high-precision method of the numerical integration HIPPI (HIgh Precision Polynomial Interpolation) (Eroshkin et al., 2000) was applied.
The compilation of semi-analytical theory of the Moon rotation (SMR):
1. Cassini’s relations:
a) The Moon rotates with a constant angular velocity around its polar axis. The rotation period is equal to the mean sidereal period of its orbital motion around the Earth.
b) The inclination of the lunar equator to the ecliptic is a constant angle (near 1° 32’).
c) The ascending node of the lunar orbit on the ecliptic coincides with the descending node of the lunar equator on the ecliptic.
2. The semi-analytical solutions of the lunar physical libration problem:
The compilation of semi-analytical theory of the Moon rotation (SMR):
1. Cassini’s relations:

   a) $\varphi_d + \psi_d = 180^\circ + L$
   b) $\theta_d = I$
   c) $\psi_d = \Omega$

2. The semi-analytical solutions of the lunar physical libration problem:
   a) MP500 is the 3rd degree solution for the force function ($C_{j0}^*$ for $j=2,3$, $C_{2k}^*$, $C_{3k}^*$, $S_{3k}^*$ for $k=1,2,3$). (Moons M., 1982),
   b) Additional solution for the 4th degree perturbations of the force function ($C_{40}^*$, $C_{4k}^*$, $S_{4k}^*$ for $k=1,\ldots,4$). (Eckhardt D.H., 1981),
   c) Effect of planetary perturbations (Moons M., 1984),
   d) Effect of the Earth's flattening (Pešek I., 1982).
The physical libration of the Moon:
Since Cassini’s relations are not exact, one must consider the perturbed Euler angles:

\[
\begin{align*}
\text{a)} \quad & \psi_d + \varphi_d = 180^\circ + L + \tau \\
\text{b)} \quad & \theta_d = I + \rho \\
\text{c)} \quad & \psi_d = \Omega + \sigma
\end{align*}
\]

where \(\psi_d\) is the longitude of the descending node of date of the lunar equator,
\(\theta_d\) is the inclination of the lunar equator to the ecliptic of date,
\(\varphi_d\) is the proper rotation angle between the descending node of date and the principal axis \(A\) (with the minimum moment of inertia);
\(L\) is the mean longitude of the Moon and
\(\Omega\) is the mean longitude of the ascending node of its orbit;
\(\tau, \rho\) and \(\sigma\) are the perturbing terms of the physical librations in the longitude, in the inclination and in the node longitude, respectively.
Fig. 1. Triangle defines the reduction of Euler angles from ecliptic of date to the fixed ecliptic of epoch J2000 for the Moon rotation.
Reduction formulae of Euler angles from the ecliptic of date
to the fixed ecliptic of epoch J2000 for the Moon rotation.

\[
\cos \theta = \cos \theta_d \cos \pi + \sin \theta_d \sin \pi \cos (\Pi - \psi_d)
\]

\[
\sin (\varphi - \varphi_d) = \frac{\sin \pi \sin (\Pi - \psi_d)}{\sin \theta}
\]

\[
\sin (\psi - \psi_d) = -\frac{\cos \theta + \cos \theta_d \sin (\varphi - \varphi_d)}{1 + \cos \pi}
\]

\[
\dot{\theta} = \dot{\theta}_d \cos (\varphi - \varphi_d) - \dot{\pi} \cos (\Pi - \psi) + (\dot{\Pi} - \dot{\psi}_d) \sin \pi \sin (\Pi - \psi)
\]

\[
\dot{\varphi} - \dot{\varphi}_d = \frac{(\dot{\Pi} - \dot{\psi}_d) \sin \pi \cos (\Pi - \psi) + \dot{\pi} \sin (\Pi - \psi) - \dot{\theta}_d \sin (\varphi - \varphi_d) \cos \theta}{\sin \theta}
\]

\[
\dot{\Pi} - \dot{\psi} = \frac{(\dot{\Pi} - \dot{\psi}_d) \sin \theta_d \cos (\varphi - \varphi_d) + \dot{\pi} \sin (\Pi - \psi) \cos \theta - \dot{\theta}_d \sin (\varphi - \varphi_d)}{\sin \theta}
\]

\[
\theta \quad \text{– the inclination of the lunar equator to the fixed ecliptic J2000;}
\]

\[
\psi \quad \text{– the longitude of the descending node of epoch J2000 of the lunar equator;}
\]

\[
\varphi \quad \text{– the proper rotation angle between the descending node of epoch J2000 and the principal axis A (with the minimum moment of inertia).}
\]
Fig. 2. Numerical solution minus SMR solution for the perturbing terms of the Lunar physical librations.

The graph shows the variations in ΔIσ, Δρ, and Δτ over years from 1750 to 2169. The y-axis represents arc seconds with values ranging from -1.5 to 0.5.
Spectral analysis (SA) method (ALGORITHM-A) for removal of the periodical terms from the discrepancies

Discrepancies: numerical solutions minus SMR

Set of libration's terms of SMR

**LSQ method** computes the amplitudes of the spectrum of discrepancies

**LSQ method** determines the amplitude and phase of the largest harmonic

if $|A_m| > |\sigma|$ 

No

Yes

**Computation of the new libration's term**

Removal of this harmonic till the end of spectrum from discrepancies and spectrum

libration's terms of SMR

Construction of the new libration's series MRS2010a
Spectral analysis (SA) method (A L G O R I T H M – A) for removal of the periodical terms from the discrepancies

Discrepancies: numerical solutions minus SMR

Set of libration's terms of SMR

**LSQ method** computes the amplitudes of the spectrum of discrepancies

**LSQ method** determines the amplitude and phase of the largest harmonic

if \( |A_m| > |\sigma| \)

No

Yes...

Construction of the new libration's series MRS2010a

Removal of this harmonic from discrepancies and spectrum

till the end of spectrum
Spectral analysis (SA) method (ALGORITHM A) for removal of the periodical terms from the discrepancies

Discrepancies: numerical solutions minus SMR

Set of libration's terms of SMR

**LSQ method** computes the amplitudes of the spectrum of discrepancies

**LSQ method** determines the amplitude and phase of the largest harmonic

if |Am| > |σ|

... Construction of the new libration's series MRS2010a

Yes

Removal of this harmonic from discrepancies and spectrum

No

till the end of spectrum
Spectral analysis (SA) method (АЛГОРИТМ –B) for removal of the periodical terms from the discrepancies

Discrepancies: numerical solutions minus SMR

Set of libration's terms of SMR

**LSQ method** computes the amplitudes of the spectrum of discrepancies

**LSQ method** determines the amplitude and phase of the largest harmonic

if $|Am| > |\sigma|$  

No  

Yes

... Construction of the new libration's series MRS2010

Removal of this harmonic from discrepancies and spectrum

till the end of spectrum

till the end of the set
Fig. 3. Spectrum of the discrepancies between numerical and semi-analytical (SMR) solutions for $\tau$. 

![Graph showing spectrum of discrepancies]
Fig. 4. Spectrum of the discrepancies between numerical and semi-analytical (SMR) solutions for $\tau$. 

![Spectrum of discrepancies between numerical and semi-analytical solutions for $\tau$.](image)
Fig. 5. Residuals after the formal removal of the 351 periodical terms from the discrepancies between numerical and semi-analytical (SMR) solutions.
Fig. 6. Residuals after the formal removal of the 351 periodical terms from the discrepancies between numerical and semi-analytical (SMR) solutions.
Fig. 7. Numerical solution 2 minus MRS2010 solution

Numerical solution 2a minus MRS2010a solution

ALGORITHM – A
ALGORITHM – B

ΔІσ
Δρ
Δτ
YEARS
mas
Fig. 8. Numerical solution *minus* SMR solution

Numerical solution 2 *minus* MRS2010 solution

![Graph showing ΔIσ, Δρ, Δτ as functions of years.]
Fig. 8. Numerical solution 2 minus MRS2010 solution (zoom)

A L G O R I T H M – B

\[ \Delta \sigma, \Delta \rho, \Delta \tau \]
Summary

1. The appearance of the harmonics of the fictive free physical librations in the numerical solution is explained by errors of the initial conditions, which are defined by not enough precise semi-analytical solutions. The periods of the harmonics of the free librations are equal 2.9 years for $\Delta \tau$, and 27.2 days, 24 years for $\Delta \rho$ and $\Delta I\sigma$.

2. We demonstrated that the ALGORITHM –B (where the spectrum is constructed anew after every removal of the largest of the residual harmonic from the discrepancies) is more accurate than the ALGORITHM –A (where the spectrum is constructed only one time).

3. The ALGORITHM –B, is very time-consuming hence not suitable for the investigation of the long time series (for example, our previous investigation the Earth rotation (V.V.Pashkevich and G.I.Eroshkin, Proceedings “Journees 2005”, 2005) ), but the ALGORITHM –B is very well suitable for the investigation of the small time series (the Moon rotation).
CONCLUSION

• The ALGORITHM –B is the optimal scheme for the research and elaboration of the numerical model of the Moon rotation.

• The new high-precision the Moon rotation series (MRS2010), dynamically adequate to the DE200/LE200 ephemeris over 400 year time interval, was constructed. MRS2010 include about 1520 periodical terms.

• Discrepancies between the numerical solution and MRS2010 do not surpass 20 mas over 400 year time interval.
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