POST-POST-NEWTONIAN LIGHT PROPAGATION WITHOUT INTEGRATING THE GEODESIC EQUATIONS

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ABSTRACT. A new derivation of the propagation direction of light is given for a 3-parameter family of static, spherically symmetric space-times within the post-post-Newtonian framework. The emitter and the observer are both located at a finite distance. The case of a ray emitted at infinity is also treated.

1. INTRODUCTION

The aim of this work is to present a new calculation of the propagation direction of light rays in a 3-parameter family of static, spherically symmetric space-times within the post-post-Newtonian framework. Rather than deriving the results from an integration of the geodesic equations, we obtain the desired expressions by a straightforward differentiation of the time delay function (see, e. g., Teyssandier & Le Poncin-Lafitte 2008 and Refs. therein). This study is motivated by the fact that any in-depth discussion of the highest accuracy tests of gravitational theories requires to evaluate the corrections of order higher than one in powers of the Schwarzschild radius (see, e.g., Ashby & Bertotti 2010 for the Cassini experiment). Even for the Gaia mission, a discrepancy between the analytical post-Newtonian solution and a computational estimate has recently necessitated a thorough analysis of the post-post-Newtonian propagation of light (see Klioner & Zschocke 2010 and Refs. therein).

2. LIGHT DIRECTION IN SPHERICALLY SYMMETRIC SPACE-TIMES

The gravitational field is assumed to be generated by an isolated spherically symmetric body of mass M. Setting $m = GM/c^2$, the metric is supposed to be of the form

$$ds^2 = \left(1 - \frac{2m}{r} + 2\beta\frac{m^2}{r^2} + \cdots\right)\left(dx^0\right)^2 - \left(1 + 2\gamma\frac{m}{r} + \frac{3}{2}\epsilon\frac{m^2}{r^2} + \cdots\right)\delta_{ij}dx^i dx^j, \tag{1}$$

where $r = \sqrt{\delta_{ij}x^i x^j}$, β and γ are the usual post-Newtonian parameters, and ϵ is a post-post-Newtonian parameter ($\beta = \gamma = \epsilon = 1$ in general relativity). We put $x^0 = ct$ and $\boldsymbol{x} = (x^i)$, with $i = 1, 2, 3$.

Consider a photon emitted at a point $\boldsymbol{x}_A$ at an instant t_A and received at a point $\boldsymbol{x}_B$ at an instant t_B. The propagation direction of this photon at any point x of its path is characterized by the triple

$$\widehat{\underline{\boldsymbol{l}}} = (l_i/l_0) = (l_1/l_0, l_2/l_0, l_3/l_0)\,, \tag{2}$$

where l_0 and l_i are the covariant components of the vector tangent to the ray, i.e. the quantities defined by $l_\alpha = g_{\alpha\beta}dx^\beta/d\lambda$, $g_{\alpha\beta}$ denoting the metric components and λ an arbitrary parameter along the ray.

Denote by $\widehat{\underline{\boldsymbol{l}}}_A$ and $\widehat{\underline{\boldsymbol{l}}}_B$ the expressions of $\widehat{\underline{\boldsymbol{l}}}$ at points $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, respectively. In any stationary space-time, these triples can be derived from the relations (see Le Poncin-Lafitte et al. 2004)

$$\left(\frac{l_i}{l_0}\right)_A = c\frac{\partial \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)}{\partial x^i_A}, \qquad \left(\frac{l_i}{l_0}\right)_B = -c\frac{\partial \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)}{\partial x^i_B}, \tag{3}$$

where $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ is the expression giving the travel time of a photon as a function of $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$:

$$t_B - t_A = \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B). \tag{4}$$

For the metric (1), $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ is given by (see, e.g., Teyssandier & Le Poncin-Lafitte 2008):

$$\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{|\boldsymbol{x}_B - \boldsymbol{x}_A|}{c} + \frac{(\gamma+1)m}{c} \ln\left(\frac{r_A + r_B + |\boldsymbol{x}_B - \boldsymbol{x}_A|}{r_A + r_B - |\boldsymbol{x}_B - \boldsymbol{x}_A|}\right)$$
$$+ m^2 \frac{|\boldsymbol{x}_B - \boldsymbol{x}_A|}{c}\left[\kappa \frac{\arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{x}_A \times \boldsymbol{x}_B|} - \frac{(\gamma+1)^2}{r_A r_B + \boldsymbol{x}_A.\boldsymbol{x}_B}\right] + \cdots, \tag{5}$$

where

$$\boldsymbol{n}_A = \frac{\boldsymbol{x}_A}{r_A}, \quad \boldsymbol{n}_B = \frac{\boldsymbol{x}_B}{r_B}, \quad \kappa = \frac{8 - 4\beta + 8\gamma + 3\epsilon}{4}. \tag{6}$$

Substituting for $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ from Eq. (5) into Eqs. (3) yields $\widehat{\underline{\boldsymbol{l}}}_A$ and $\widehat{\underline{\boldsymbol{l}}}_B$ as linear combinations of $\boldsymbol{n}_A$ and $\boldsymbol{n}_B$. However, it is more convenient to introduce the unit vector $\boldsymbol{N}_{AB}$ defined by

$$\boldsymbol{N}_{AB} = \frac{\boldsymbol{x}_B - \boldsymbol{x}_A}{|\boldsymbol{x}_B - \boldsymbol{x}_A|} \tag{7}$$

and the unit vector $\boldsymbol{P}_{AB}$ orthogonal to $\boldsymbol{N}_{AB}$ defined as $\boldsymbol{OH}/|\boldsymbol{OH}|$, H being the orthogonal projection of the center O of the mass M on the straight line passing through $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, that is

$$\boldsymbol{P}_{AB} = \boldsymbol{N}_{AB} \times \left(\frac{\boldsymbol{n}_A \times \boldsymbol{n}_B}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\right). \tag{8}$$

Using Eqs. (5)-(8), we deduce the following proposition from Eqs. (3).

Proposition 1. *The triples $\widehat{\underline{\boldsymbol{l}}}_A$ and $\widehat{\underline{\boldsymbol{l}}}_B$ are given by*

$$\widehat{\underline{\boldsymbol{l}}}_A = -\boldsymbol{N}_{AB} - \frac{m}{r_A}\left\{(\gamma+1) + \frac{m}{r_A}\left[\kappa - \frac{(\gamma+1)^2}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\right]\right\}\boldsymbol{N}_{AB}$$
$$- \frac{m}{r_A}\left\{(\gamma+1)\frac{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B} + \frac{m}{r_A}\frac{1}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\left\{\kappa\left[\frac{\arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\left(1 - \frac{r_A}{r_B}\boldsymbol{n}_A.\boldsymbol{n}_B\right)\right.\right.\right.$$
$$\left.\left.\left. + \frac{r_A}{r_B} - \boldsymbol{n}_A.\boldsymbol{n}_B\right] - (\gamma+1)^2\left(1 + \frac{r_A}{r_B}\right)\frac{1 - \boldsymbol{n}_A.\boldsymbol{n}_B}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\right\}\right\}\boldsymbol{P}_{AB} \tag{9}$$

and

$$\widehat{\underline{\boldsymbol{l}}}_B = -\boldsymbol{N}_{AB} - \frac{m}{r_B}\left\{\gamma + 1 + \frac{m}{r_B}\left[\kappa - \frac{(\gamma+1)^2}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\right]\right\}\boldsymbol{N}_{AB}$$
$$+ \frac{m}{r_B}\left\{(\gamma+1)\frac{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B} + \frac{m}{r_B}\frac{1}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\left\{\kappa\left[\frac{\arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\left(1 - \frac{r_B}{r_A}\boldsymbol{n}_A.\boldsymbol{n}_B\right)\right.\right.\right.$$
$$\left.\left.\left. + \frac{r_B}{r_A} - \boldsymbol{n}_A.\boldsymbol{n}_B\right] - (\gamma+1)^2\left(1 + \frac{r_B}{r_A}\right)\frac{1 - \boldsymbol{n}_A.\boldsymbol{n}_B}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\right\}\right\}\boldsymbol{P}_{AB}, \tag{10}$$

respectively.

In any static, spherically symmetric space-time the geodesic equations imply that the vector $\boldsymbol{L}$ defined as $\boldsymbol{L} = -\boldsymbol{x} \times \widehat{\underline{\boldsymbol{l}}}$ is a constant of the motion. The null geodesics considered here are assumed to be unbound. Consequently the magnitude of $\boldsymbol{L}$ is such that $|\boldsymbol{L}| = lim_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x} \times d\boldsymbol{x}/cdt|$ since $\widehat{\underline{\boldsymbol{l}}} \longrightarrow -(d\boldsymbol{x}/cdt)_\infty$ when $|\boldsymbol{x}| \longrightarrow \infty$. So the quantity b defined by

$$b = |-\boldsymbol{x} \times \widehat{\underline{\boldsymbol{l}}}| \tag{11}$$

is the Euclidean distance between the asymptote to the ray and the line parallel to this asymptote passing through the center O as measured by an inertial observer at rest at infinity. Hence b may be considered as *the impact parameter* of the ray (see, e.g., Chandrasekhar 1983). Besides its geometric meaning, b presents the interest to be *intrinsic*, since it corresponds to a quantity which could be really measured.

Substituting for $\underline{\widehat{\boldsymbol{l}}}_B$ from Eq. (10) into Eq. (11), introducing the zeroth-order distance of closest approach r_c defined as

$$r_c = \frac{r_A r_B}{|\boldsymbol{x}_B - \boldsymbol{x}_A|}|\boldsymbol{n}_A \times \boldsymbol{n}_B|, \tag{12}$$

and then using $(r_A + r_B)|\boldsymbol{n}_A \times \boldsymbol{n}_B|/|\boldsymbol{x}_B - \boldsymbol{x}_A| = |\boldsymbol{N}_{AB} \times \boldsymbol{n}_A| + |\boldsymbol{N}_{AB} \times \boldsymbol{n}_B|$, we get

$$b = r_c\left[1 + \frac{(\gamma+1)m}{r_c}\frac{|\boldsymbol{N}_{AB} \times \boldsymbol{n}_A| + |\boldsymbol{N}_{AB} \times \boldsymbol{n}_B|}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B} + \cdots\right]. \tag{13}$$

Using this expansion of b, we obtain the proposition which follows.

Proposition 2. *In terms of the impact parameter b, the triples $\underline{\widehat{\boldsymbol{l}}}_A$ and $\underline{\widehat{\boldsymbol{l}}}_B$ may be written as*

$$\begin{aligned}\underline{\widehat{\boldsymbol{l}}}_A = -\boldsymbol{N}_{AB} - \frac{m|\boldsymbol{N}_{AB} \times \boldsymbol{n}_A|}{b}\left\{\gamma + 1 + \frac{m}{b}\left[\kappa|\boldsymbol{N}_{AB} \times \boldsymbol{n}_A| + (\gamma+1)^2\frac{|\boldsymbol{N}_{AB} \times \boldsymbol{n}_B|}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\right]\right\}\boldsymbol{N}_{AB} \\ -\frac{m|\boldsymbol{N}_{AB} \times \boldsymbol{n}_A|}{b}\left\{(\gamma+1)\frac{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\right. \\ \left. + \frac{\kappa m}{b}\left[\frac{\arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\boldsymbol{N}_{AB}.\boldsymbol{n}_B - \boldsymbol{N}_{AB}.\boldsymbol{n}_A\right]\right\}\boldsymbol{P}_{AB},\end{aligned} \tag{14}$$

$$\begin{aligned}\underline{\widehat{\boldsymbol{l}}}_B = -\boldsymbol{N}_{AB} - \frac{m|\boldsymbol{N}_{AB} \times \boldsymbol{n}_B|}{b}\left\{\gamma + 1 + \frac{m}{b}\left[\kappa|\boldsymbol{N}_{AB} \times \boldsymbol{n}_B| + (\gamma+1)^2\frac{|\boldsymbol{N}_{AB} \times \boldsymbol{n}_A|}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\right]\right\}\boldsymbol{N}_{AB} \\ +\frac{m|\boldsymbol{N}_{AB} \times \boldsymbol{n}_B|}{b}\left\{(\gamma+1)\frac{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\right. \\ \left. - \frac{\kappa m}{b}\left[\frac{\arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\boldsymbol{N}_{AB}.\boldsymbol{n}_A - \boldsymbol{N}_{AB}.\boldsymbol{n}_B\right]\right\}\boldsymbol{P}_{AB}.\end{aligned} \tag{15}$$

3. DEFLECTION OF A LIGHT RAY EMITTED AT INFINITY

Assume now that the ray arriving at $\boldsymbol{x}_B$ is emitted at infinity in a direction defined by a unit vector $\boldsymbol{N}_e$. Substituting $\boldsymbol{N}_e$ for $\boldsymbol{N}_{AB}$ and $-\boldsymbol{N}_e$ for $\boldsymbol{n}_A$ in Eq. (15) yields the expression of $\widehat{\boldsymbol{l}}_B$, where b is furnished by the limit of Eqs. (12) and (13) when $r_A \to \infty$ and $\boldsymbol{n}_A \to -\boldsymbol{N}_e$. We can set a proposition as follows.

Proposition 3. *For a light ray emitted at infinity in a direction $\boldsymbol{N}_e$ and arriving at $\boldsymbol{x}_B$, $\underline{\widehat{\boldsymbol{l}}}_B$ is given by*

$$\begin{aligned}\underline{\widehat{\boldsymbol{l}}}_B = -\boldsymbol{N}_e - \frac{m|\boldsymbol{N}_e \times \boldsymbol{n}_B|}{b}\left[\gamma + 1 + \frac{\kappa m|\boldsymbol{N}_e \times \boldsymbol{n}_B|}{b}\right]\boldsymbol{N}_e \\ +\frac{m}{b}\left\{(\gamma+1)(1 + \boldsymbol{N}_e.\boldsymbol{n}_B) + \frac{\kappa m}{b}\left[\pi - \arccos(\boldsymbol{N}_e.\boldsymbol{n}_B)\right.\right. \\ \left.\left. + |\boldsymbol{N}_e \times \boldsymbol{n}_B|\boldsymbol{N}_e.\boldsymbol{n}_B\right]\right\}\boldsymbol{P}_B(\boldsymbol{N}_e),\end{aligned} \tag{16}$$

where $\boldsymbol{P}_B(\boldsymbol{N}_e)$ is the unit vector orthogonal to $\boldsymbol{N}_e$ defined as

$$\boldsymbol{P}_B(\boldsymbol{N}_e) = -\boldsymbol{N}_e \times \frac{\boldsymbol{N}_e \times \boldsymbol{n}_B}{|\boldsymbol{N}_e \times \boldsymbol{n}_B|} \tag{17}$$

and b is the impact parameter of the ray, namely

$$b = r_c\left[1 + \frac{(\gamma+1)m}{r_c}\frac{|\boldsymbol{N}_e \times \boldsymbol{n}_B|}{1 - \boldsymbol{N}_e.\boldsymbol{n}_B} + \cdots\right], \tag{18}$$

with $r_c = r_B|\boldsymbol{N}_e \times \boldsymbol{n}_B|$.

The deflection of the ray at point $\boldsymbol{x}_B$ may be characterized by the angle $\Delta\chi_B$ made by the vector $\boldsymbol{N}_e$ and a vector tangent to the ray at $\boldsymbol{x}_B$. We have

$$\Delta\chi_B = \frac{|\boldsymbol{N}_e \times \widehat{\boldsymbol{\underline{l}}}_B|}{|\widehat{\boldsymbol{\underline{l}}}_B|} + O(1/c^6). \tag{19}$$

Substituting for $\widehat{\boldsymbol{\underline{l}}}_B$ from Eq. (16) into Eq. (19), and then introducing the angle ϕ_B between $\boldsymbol{N}_e$ and $\boldsymbol{n}_B$ defined by

$$\boldsymbol{N}_e.\boldsymbol{n}_B = \cos\phi_B, \qquad 0 \leq \phi_B \leq \pi, \tag{20}$$

we get

$$\Delta\chi_B = \frac{(\gamma+1)GM}{c^2 b}(1+\cos\phi_B) + \frac{G^2M^2}{c^4b^2}\left[\kappa\left(\pi-\phi_B+\frac{1}{2}\sin 2\phi_B\right)-(\gamma+1)^2(1+\cos\phi_B)\sin\phi_B\right], \tag{21}$$

where the impact parameter given by Eq. (18) may be rewritten as

$$b = r_c\left[1+\frac{(\gamma+1)GM}{c^2r_c}\frac{\sin\phi_B}{1-\cos\phi_B}+\cdots\right], \qquad r_c = r_B\sin\phi_B. \tag{22}$$

It may be seen from the formulas given in Teyssandier & Le Poncin-Lafitte 2006 that $\phi_B + \Delta\chi_B$ is the angular distance between the center O and the source at infinity as measured at $\boldsymbol{x}_B$ by a static observer, i.e. an observer at rest with respect to the coordinates x^i. It will be shown in a subsequent paper that this property implies that $\Delta\chi_B$ can be regarded as an *intrinsic* quantity.

The $1/c^2$ term in Eq. (21) is currently used in VLBI astrometry. If b is replaced by its coordinate expression (22), it may be seen that $\Delta\chi_B$ is given by an expression as follows

$$\Delta\chi_B = \frac{(\gamma+1)GM}{c^2r_c}(1+\cos\phi_B) + \frac{G^2M^2}{c^4r_c^2}\left[\kappa\left(\pi-\phi_B+\frac{1}{2}\sin 2\phi_B\right)-(\gamma+1)^2(1+\cos\phi_B)\sin\phi_B \right. \\ \left. \underbrace{-(\gamma+1)^2\frac{(1+\cos\phi_B)^2}{\sin\phi_B}}\right]. \tag{23}$$

For a ray grazing a mass M of radius r_0, the underbraced term in the r.h.s. of Eq. (23) generates a post-post-Newtonian contribution $(\Delta\chi_B^{(2)})_{grazing} \approx -4(\gamma+1)^2(GM/c^2r_0)^2(r_B/r_0)$ which can be great if $r_B \gg r_0$. For Jupiter, $(\Delta\chi_B^{(2)})_{grazing} = 16.1\,\mu$as if the observer is located at a distance from Jupiter $r_B = 6$ AU: this value is appreciably greater than the level of accuracy expected for Gaia. However, this 'enhanced' term is due to the use of the coordinate-dependent quantity r_c instead of the intrinsic impact parameter b. This result confirms the conclusion recently drawn in Klioner & Zschocke 2010.

4. CONCLUSION

Deriving the second-order terms in the propagation direction of light from the time transfer function rather than from the null geodesic equations is a very elegant and powerful procedure. The application of this method to a ray emitted at infinity and received by a static observer located at a finite distance from the central mass is easy and yields an intrinsic characterization of the gravitational bending of light.

5. REFERENCES

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