SOME RECENT DEVELOPMENTS IN RELATIVISTIC MODELING OF TIME AND FREQUENCY TRANSFERS

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ABSTRACT. We determine the relativistic effects of the tidal potentials on the time and frequency transfers between an atomic clock orbiting round the Earth and a ground clock. These effects are estimated for ESA Atomic Clocks Ensemble in Space (ACES) mission planned to be launched in 2012.

1. INTRODUCTION

The ACES mission is planned to compare a cold atom clock PHARAO onboard the ISS with terrestrial clocks by the mean of a microwave link. PHARAO is expected to reach a frequency stability of $1 \cdot 10^{-16}$ for an integration time of ten days, with a relative accuracy of $1 \cdot 10^{-16}$ (see Blanchet et al. 2001 and Duchayne et al. 2007). A primary objective of ACES will be to compare ground clocks in common view below the $1 \cdot 10^{-17}$ level after one day of integration. This level of performance requires to determine the influence of the tidal potentials on the frequency shifts in the vicinity of the Earth.

2. GENERAL FORMULA GIVING THE FREQUENCY SHIFT

We assume that space-time can be covered by a global coordinate system $x^\alpha = (x^0, x^i) \equiv (ct, \boldsymbol{x})$, in which the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

is such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1). \tag{2}$$

The coordinate travel time $t_B - t_A$ of a photon between an emission point $x_A = (ct_A, \boldsymbol{x}_A)$ and a reception point $x_B = (ct_B, \boldsymbol{x}_B)$ may be considered as a function of $\boldsymbol{x}_A, t_B$ and $\boldsymbol{x}_B$, so that we can write

$$t_B - t_A = \mathcal{T}_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B), \tag{3}$$

where $\mathcal{T}_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B)$ represents what we call the "reception time transfer function". For the decomposition of the metric given by Eq. (2), the function $c\mathcal{T}_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B)$ may be written in the form

$$c\mathcal{T}_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) = |\boldsymbol{x}_B - \boldsymbol{x}_A| + \Delta_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B), \tag{4}$$

where $\Delta_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B)$ is of the order of the gravitational perturbation $h_{\mu\nu}$.

The knowledge of $\Delta_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B)$ enables to treat both the problems of time transfers and of frequency transfers. Indeed, consider a light signal emitted at point x_A by an observer A moving with a unit 4-velocity vector $u_A^\alpha = dx_A^\alpha/ds_A$ and received at point x_B by an observer B moving with a unit 4-velocity vector $u_B^\alpha = dx_B^\alpha/ds_B$. Let ν_A be the frequency of the signal as measured by A at x_A and ν_B the frequency of the signal as measured by B at x_B. A simple reasoning shows that within the geometric optics approximation the frequency shift between x_A and x_B is given by the formula

$$\frac{\nu_A}{\nu_B} = \frac{u_A^0}{u_B^0} \frac{1 - \boldsymbol{N}_{AB} \cdot \dfrac{\boldsymbol{v}_A}{c} + \dfrac{\partial \Delta_r}{\partial x_A^i} \dfrac{v_A^i}{c}}{1 - \boldsymbol{N}_{AB} \cdot \dfrac{\boldsymbol{v}_B}{c} - \dfrac{\partial \Delta_r}{c \partial t_B} - \dfrac{\partial \Delta_r}{\partial x_B^j} \dfrac{v_B^j}{c}}, \qquad \boldsymbol{N}_{AB} = \frac{\boldsymbol{x}_B - \boldsymbol{x}_A}{|\boldsymbol{x}_B - \boldsymbol{x}_A|}, \tag{5}$$

where $\boldsymbol{v}_A = (d\boldsymbol{x}/dt)_A$ and $\boldsymbol{v}_B = (d\boldsymbol{x}/dt)_B$ are the coordinate velocities of observers A and B, respectively. It follows from $g_{\mu\nu} u^\mu u^\nu = 1$ that u_A^0 and u_B^0 may be calculated by the relation

$$u^0 = \frac{dx^0}{ds} = \left[1 + h_{00} + 2h_{0i}\frac{v^i}{c} - (\delta_{ij} - h_{ij})\frac{v^i v^j}{c^2}\right]^{-1/2} \tag{6}$$

applied to the observers A and B, respectively.

The formula (5) shows that it is sufficient to determine $\Delta_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B)$ at the order $1/c^2$ when the frequency shift is required at the order $1/c^3$.

3. EFFECT OF THE TIDAL POTENTIALS

Since we are concerned here by time and frequency transfers between a satellite of the Earth and a ground station, we now suppose that the coordinate system $(ct, \boldsymbol{x})$ constitutes a local nonrotating geocentric reference system (GCRS). Then it may be assumed with a sufficient approximation that the gravitational potentials are given by (see Klioner & Soffel 2000):

$$G_{00} = 1 - \frac{2}{c^2}W + O(1/c^4), \tag{7}$$

$$G_{0i} = O(1/c^3), \tag{8}$$

$$G_{ij} = -\left(1 + \frac{2\gamma}{c^2}W\right)\delta_{ij} + O(1/c^4), \tag{9}$$

where γ is the well-known post-Newtonian parameter involved in light deflection ($\gamma = 1$ in general relativity) and W may be decomposed as

$$W(t, \boldsymbol{x}) = W_\oplus(t, \boldsymbol{x}) + W^{(T)}(t, \boldsymbol{x}) + Q_i x^i + O(1/c^2), \tag{10}$$

where $W_\oplus(t, \boldsymbol{x})$ is the potential of the Earth , $W^{(T)}(t, \boldsymbol{x})$ is the tidal potential and Q_i is the non geodesic acceleration of the Earth center of mass with respect to the GCRS. At the order $O(1/c^3)$, the ratio ν_A/ν_B is given by

$$\frac{\nu_A}{\nu_B} = \frac{1 - \boldsymbol{N}_{AB}\cdot\frac{\boldsymbol{v}_A}{c}}{1 - \boldsymbol{N}_{AB}\cdot\frac{\boldsymbol{v}_B}{c}}\left(1 + \frac{v_A^2 - v_B^2}{2c^2}\right) + \left(\frac{\delta\nu}{\nu}\right)_g, \tag{11}$$

where $(\delta\nu/\nu)_g$ contains all the contributions of the gravitational field:

$$\left(\frac{\delta\nu}{\nu}\right)_g = \frac{1}{c^2}(W_A - W_B)\left[1 - \frac{1}{c}\boldsymbol{N}_{AB}.(\boldsymbol{v}_A - \boldsymbol{v}_B)\right]$$
$$+ \frac{\partial\Delta_r}{\partial x_A^i}\frac{v_A^i}{c} + \frac{\partial\Delta_r}{c\partial t_B} + \frac{\partial\Delta_r}{\partial x_B^j}\frac{v_B^j}{c} + O(c^{-4}). \tag{12}$$

It follows from Eq. (8) that the contributions of G_{0i} may be neglected. So the expression of $\Delta_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B)$ reduces to (see Linet & Teyssandier 2002):

$$\Delta_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) = \frac{1}{c^2}(\gamma + 1)|\boldsymbol{x}_B - \boldsymbol{x}_A|\int_0^1 W(z_-^\alpha(\lambda))d\lambda + O(1/c^3), \tag{13}$$

the integral being taken along the null straight line in Minkowski space-time defined by the parametric equations

$$z_-^0(\lambda) = -\lambda|\boldsymbol{x}_B - \boldsymbol{x}_A| + x_B^0, \quad z_-^i(\lambda) = -\lambda(x_B^i - x_A^i) + x_B^i.$$

At the order $1/c^3$, the potentials $W_\oplus$ and $W^{(T)}(t, \boldsymbol{x})$ involved in Eqs. (12) and (13) may be replaced by their respective Newtonian expressions. Moreover the contribution of the non geodesic acceleration to $(W_A - W_B)/c^2$ is completely negligible since $|c^{-2}Q_i x^i| < 10^{-20}$.

The contributions of $W_\oplus(t, \boldsymbol{x})$ were analyzed in Blanchet et al. (2001), Duchayne et al. (2007), Le Poncin-Lafitte & Lambert (2007), and Linet & Teyssandier (2002). So we discuss only the contributions of the tidal potentials. The dominant effects are due to the Moon (☾) and to the Sun ($\odot$). Putting

$$h = r_B - r_A, \quad \zeta_A = \frac{h}{r_A}, \quad \kappa_A = \frac{|\boldsymbol{x}_B - \boldsymbol{x}_A|}{r_A}, \tag{14}$$

we get to a sufficient approximation:

$$\left(\frac{\delta\nu}{\nu}\right)_{g,T} \approx \sum_{K=\leftmoon,\odot} \frac{GM_K}{2c^2 D_K}\frac{r_A^2}{D_K^2}\{\zeta_A(2+\zeta_A)$$
$$-3\kappa_A(\boldsymbol{d}_K\cdot\boldsymbol{N}_{AB})[2(\boldsymbol{d}_K\cdot\boldsymbol{n}_A)+\kappa_A(\boldsymbol{d}_K\cdot\boldsymbol{N}_{AB})]\}\,, \tag{15}$$

where

$$\boldsymbol{n}_A = \frac{\boldsymbol{x}_A}{r_A}, \quad \boldsymbol{n}_B = \frac{\boldsymbol{x}_B}{r_B}, \quad \boldsymbol{d}_K = \frac{\boldsymbol{D}_K}{D_K}, \tag{16}$$

$\boldsymbol{D}_K$ being the vector position of the body K exerting a tidal influence. Since they are slowly varying with respect to time, the values of $\boldsymbol{D}_{\leftmoon}$ and $\boldsymbol{D}_\odot$ may be taken at any instant in the range $t_A \leq t \leq t_B$.

For the ACES mission, we have $h \approx 400$ km. So $\zeta_A \approx 0.063$. Moreover, we may admit that a comparison of clocks will be acceptable only when the elevation angle of the ISS over the horizon of the ground station will be greater than 20°. Then, $0.063 \leq \kappa_A \leq 0.154$. Under these conditions, Eq. (15) leads to inequalities as follow

$$\left|\left(\frac{\delta\nu}{\nu}\right)_{g,T}\right|_{\leftmoon} \leq 2.6\cdot 10^{-17}, \qquad \left|\left(\frac{\delta\nu}{\nu}\right)_{g,T}\right|_{\odot} \leq 9.6\cdot 10^{-18}. \tag{17}$$

We conclude that the influence of the tidal potentials will be negligible in the comparison of PHARAO with a ground clock. However, a more detailed discussion will be necessary for the comparison of ground clocks in common view.

4. REFERENCES

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