CANONICAL ELEMENTS OF ROTATIONAL MOTION

T. FUKUSHIMA
National Astronomical Observatory of Japan
2-21-1, Ohsawa, Mitaka, Tokyo 181-8588, Japan
e-mail: Toshio.Fukushima@nao.ac.jp

ABSTRACT. We present a new set of canonical variables to describe general rotation of a triaxial rigid body. Explicit are both the forward and backward transformations from the new variables to the Andoyer canonical variables, which are universal. The rotational kinetic energy is expressed as a quadratic monomial of one new momentum. Consequently, the torque-free rotations are expressed as a linear function of time for the conjugate coordinate and constants of time for the rest two coordinates and three momenta. This means that the new canonical variables are universal elements in a broad sense.

1. BACKGROUND

Triggered by a recent review article on the analytical treatise on rotational motions (Gurfil et al. 2007), we initiated a series of researches on the theme. At the beginning, we intended to establish an all-purpose analytical tool such as the canonical and universal elements for the rotation of a general triaxial rigid body (Fukushima 2008d) which we summarize here. However, this was far more difficult than we expected. In order to find a clue, we changed our strategy so as to seek for a non-canonical formulation deploying the force function, U, like Lagrange's planetary equation in case of orbital motions. Again, this was too complicated to be realized.

Then, we shifted our target to a Gaussian formulation using non-canonical elements. Here the term 'Gaussian' means that not the partial derivatives of U but the torque vector components directly appear in the time evolution equations of the elements. Although we could obtain such a formulation (Fukushima 2008c), we were not fully sure of the correctness of the obtained formulation, especially whether the time evolution equation of non-canonical elements are analytically correct. A simple and independent check would be a comparison with the direct numerical integration of the equations of motion which are well established such as Euler's kinematical and dynamical equations of rotation in terms of Euler angles in the 3-1-3 convention, (ψ, θ, ϕ), and the angular velocity vector components referred to the body-fixed reference frame, $(\omega_A, \omega_B, \omega_C)$ (MacMillan 1936). We thought this an easy task. However, we immediately faced numerical difficulties in conducting simulations to examine the correctness of the analytical formulation.

The reason is clear; the kinematical equation in terms of the 3-1-3 Euler angles contain a singular point, $\theta = 0$. The singularity remains whichever convention of Euler angles we choose (Hughes 1986). Thus, we had to make an investigation on robust numerical formulations. In the course, we confirmed that Euler parameters, (q_0, q_1, q_2, q_3), are suitable substitutes to Euler angles. We succeeded to stabilize the method adopting $(q_0, q_1, q_2, q_3; \omega_A, \omega_B, \omega_C)$ as main variables by enforcing their normalization condition, $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, at every integration step. To our surprise, this method is the fastest and most precise (Fukushima 2008b).

After confirming the correctness of the Gaussian element formulation, we reformed it into a Lagrangian one by replacing the torque vector components by the corresponding partial derivatives of force function. Next we compared the obtained equation of non-canonical elements in the Lagrangian manner with those of canonical equations of motion in terms of the Andoyer variables and found the explicit relations between them. After some trials and errors of the formula manipulation of the obtained analytical relations, we arrived at a new set of canonical elements (Fukushima 2008d). However, it was difficult to prove their canonicity from the transformed relations. Thus, we searched for a suitable generating function, which is not always available for arbitrary canonical transformation. Fortunately, we could find such a function. Therefore, we will begin with the generating function.

2. A CANONICAL TRANSFORMATION OF FREEDOM TWO

Consider a following function of four variables, S, Z, ℓ, and g,

$$W(S,Z,\ell,g) \equiv Zg + \int_{\pi/2}^{\ell} \left(\sqrt{\frac{\left(a\sin^2\theta + b\cos^2\theta\right) Z^2 - aS^2}{(a-c)\sin^2\theta + (b-c)\cos^2\theta}} \right) d\theta, \tag{1}$$

where a, b, and c are arbitrary positive constants satisfying the magnitude relations; $a \geq b \geq c$ and $a > c$. Regard S and Z as the new canonical momenta and ℓ and g as the old canonical coordinates. Then, by treating W as a generating function, we obtain a new canonical transformation of freedom two;

$$(L, G; \ell, g) \rightarrow (S, Z; s, z) \tag{2}$$

where the old momenta L and G and the new coordinates s and z are determined from W (Golstein et al. 2002) as

$$L \equiv \left(\frac{\partial W}{\partial \ell}\right)_{S,Z,g}, \qquad G \equiv \left(\frac{\partial W}{\partial g}\right)_{S,Z,\ell} = Z, \qquad s \equiv \left(\frac{\partial W}{\partial S}\right)_{Z,\ell,g}, \qquad z \equiv \left(\frac{\partial W}{\partial Z}\right)_{S,\ell,g}. \tag{3}$$

Conducting these partial derivatives, we obtain the explicit form of the canonical transformation as

$$S = \left(\sqrt{\frac{d}{a}}\right) G, \;\; s = \left(\sqrt{\frac{ad}{(a-d)(b-c)}}\right) F(\varphi|m), \;\; z = g - \left(\frac{cF(\varphi|m) + (a-c)\mathit{\Pi}(\varphi, -f|m)}{\sqrt{(a-d)(b-c)}}\right). \tag{4}$$

and $Z = G$. Here $F(\varphi|m)$ and $\mathit{\Pi}(\varphi, -f|m)$ are the incomplete elliptic integrals of the first and third kind with the argument φ, the parameter m, and the characteristic $-f$ (Byrd & Friedman 1971, Wofgram 2003), and

$$d \equiv \left(a\sin^2\ell + b\cos^2\ell\right)\left(\frac{G^2 - L^2}{G^2}\right) + c\left(\frac{L^2}{G^2}\right), \qquad \varphi \equiv \frac{\pi}{2} - \ell - \tan^{-1}\left(\frac{f\sin\ell\cos\ell}{\sqrt{1+f} + 1 + f\sin^2\ell}\right),$$

$$m \equiv \frac{(a-b)(d-c)}{(a-d)(b-c)}, \qquad f \equiv \frac{a-b}{b-c}. \tag{5}$$

3. NEW CANONICAL VARIABLES OF ROTATION

Apply the transformation (2) to the Andoyer canonical variables (Andoyer 1923), $(L, G, H; \ell, g, h)$, while interpret the constants a, b, and c as the inverse of principal moments of inertia, $A \equiv I_1$, $B \equiv I_2$, and $C \equiv I_3$;

$$a = \frac{1}{A}, \qquad b = \frac{1}{B}, \qquad c = \frac{1}{C}. \tag{6}$$

Then, we obtain a new set of canonical variables to describe rotational motions, $(S, Z, H; s, z, h)$. The transformation from the Andoyer canonical variables to the new ones are explicit as provided in Equation (4). Also explicit is the transformation from the new variables to the Andoyer ones as

$$L = \left(\sqrt{\frac{a-d}{a-c}}\right) Z\mathrm{dn}(u|m), \qquad \ell = \frac{\pi}{2} - \mathrm{am}(u|m) - \tan^{-1}\left(\frac{f\mathrm{sn}(u|m)\mathrm{cn}(u|m)}{\sqrt{1+f} + 1 + f\mathrm{sn}^2(u|m)}\right),$$

$$g = z + \left(\frac{cu + (a-c)\mathrm{pn}(u, -f|m)}{\sqrt{(a-d)(b-c)}}\right), \tag{7}$$

and $G = Z$ where

$$d = a\left(\frac{S^2}{Z^2}\right), \qquad u = \left(\sqrt{\frac{(a-d)(b-c)}{ad}}\right) s, \tag{8}$$

and m is computed from this d by the second last of Equation (5). Meanwhile $\mathrm{am}(u|m)$, $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ are the Jacobian elliptic functions with the argument u and the parameter m, and

$$\mathrm{pn}(u, n|m) \equiv \mathit{\Pi}(\mathrm{am}(u|m), n|m), \tag{9}$$

is the pi amplitude function which we recently introduced (Fukushima 2008c). Using these transformation formulas, we obtain the expression of the Hamiltonian in terms of the new variables as

$$\mathcal{H} = \frac{1}{2}\left[\left(G^2 - L^2\right)\left(a\sin^2\ell + b\cos^2\ell\right) + cL^2\right] - U = \frac{1}{2}aS^2 - U, \tag{10}$$

where U is the perturbing force function. As a result, the canonical equations of motion become as simple as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \left(\frac{\partial U}{\partial s}\right)_{S,Z,H,z,h}, \qquad \frac{\mathrm{d}Z}{\mathrm{d}t} = \left(\frac{\partial U}{\partial z}\right)_{S,Z,H,s,h}, \qquad \frac{\mathrm{d}H}{\mathrm{d}t} = \left(\frac{\partial U}{\partial h}\right)_{S,Z,H,s,z},$$
$$\frac{\mathrm{d}s}{\mathrm{d}t} = aS - \left(\frac{\partial U}{\partial S}\right)_{Z,H,s,z,h}, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = -\left(\frac{\partial U}{\partial Z}\right)_{S,H,s,z,h}, \qquad \frac{\mathrm{d}h}{\mathrm{d}t} = -\left(\frac{\partial U}{\partial H}\right)_{S,Z,s,z,h}. \tag{11}$$

In case of torque-free rotations, $U = 0$. Therefore, the solutions are plainly given as

$$S = S_0, \qquad Z = Z_0, \qquad H = H_0, \qquad s = s_0 + aS_0 t, \qquad z = z_0, \qquad h = h_0, \tag{12}$$

where the quantities with the suffix 0 are their values when $t = 0$. This means that the new variables are elements in a broad sense (Fukushima 2008a). The Andoyer canonical variables are universal in the sense they can describe all three modes of rotation; the short-, the middle-, and the long-axis modes. Therefore, the new canonical elements are also universal.

4. PARTIAL DERIVATIVES

Let us describe how to evaluate the partial derivatives of the force function with respect to the new variables, $\partial U/\partial(S,Z,H,s,z,h)$. Assume that already available are the partial derivatives with respect to the Andoyer canonical variables, $\partial U/\partial(L,G,H,\ell,g,h)$, the calculation of which has been extensively discussed in the literature. Then, the resulting procedure to compute $\partial U/\partial(S,Z,H,s,z,h)$ is to give the Jacobian matrix of the transformation, namely the 6×6 matrix of partial derivatives of the Andoyer canonical variables with respect to the new ones, $\partial(L,G,H,\ell,g,h)/\partial(S,Z,H,s,z,h)$. Among 36 components of the Jacobian matrix, (1) zero are 23 of them; $(\partial L/\partial H)$, $(\partial L/\partial z)$, $(\partial L/\partial h)$, $(\partial G/\partial S)$, $(\partial G/\partial H)$, $(\partial G/\partial z)$, $(\partial G/\partial h)$, $(\partial H/\partial S)$, $(\partial H/\partial Z)$, $(\partial H/\partial s)$, $(\partial H/\partial z)$, $(\partial H/\partial h)$, $(\partial \ell/\partial H)$, $(\partial \ell/\partial z)$, $(\partial \ell/\partial h)$, $(\partial g/\partial H)$, $(\partial g/\partial z)$, $(\partial g/\partial h)$, $(\partial h/\partial H)$, $(\partial h/\partial S)$, $(\partial h/\partial Z)$, $(\partial h/\partial s)$, and $(\partial h/\partial z)$, (2) unity are four of them; $(\partial G/\partial Z)$, $(\partial g/\partial z)$, $(\partial H/\partial H)$, and $(\partial h/\partial h)$, and (3) non trivial are the rest nine components, which are explicitly expressed as

$$\left(\frac{\partial L}{\partial S}\right) = \left(\frac{\partial L}{\partial d}\right)\left(\frac{\partial d}{\partial S}\right), \qquad \left(\frac{\partial \ell}{\partial S}\right) = \left(\frac{\partial \ell}{\partial d}\right)\left(\frac{\partial d}{\partial S}\right), \qquad \left(\frac{\partial g}{\partial S}\right) = \left(\frac{\partial g}{\partial d}\right)\left(\frac{\partial d}{\partial S}\right),$$
$$\left(\frac{\partial L}{\partial Z}\right) = \left(\frac{\partial L}{\partial Z}\right)^* + \left(\frac{\partial L}{\partial d}\right)\left(\frac{\partial d}{\partial Z}\right), \qquad \left(\frac{\partial \ell}{\partial Z}\right) = \left(\frac{\partial \ell}{\partial d}\right)\left(\frac{\partial d}{\partial Z}\right), \qquad \left(\frac{\partial g}{\partial Z}\right) = \left(\frac{\partial g}{\partial d}\right)\left(\frac{\partial d}{\partial Z}\right),$$
$$\left(\frac{\partial L}{\partial s}\right) = \left(\frac{\partial L}{\partial u}\right)\left(\frac{\partial u}{\partial s}\right), \qquad \left(\frac{\partial \ell}{\partial s}\right) = \left(\frac{\partial \ell}{\partial u}\right)\left(\frac{\partial u}{\partial s}\right), \qquad \left(\frac{\partial g}{\partial s}\right) = \left(\frac{\partial g}{\partial u}\right)\left(\frac{\partial u}{\partial s}\right), \tag{13}$$

where

$$\left(\frac{\partial L}{\partial d}\right) = \left(\frac{\partial L}{\partial d}\right)^* + \left(\frac{\partial L}{\partial m}\right)\left(\frac{\partial m}{\partial d}\right) + \left(\frac{\partial L}{\partial u}\right)\left(\frac{\partial u}{\partial d}\right), \qquad \left(\frac{\partial \ell}{\partial d}\right) = \left(\frac{\partial \ell}{\partial m}\right)\left(\frac{\partial m}{\partial d}\right) + \left(\frac{\partial \ell}{\partial u}\right)\left(\frac{\partial u}{\partial d}\right),$$
$$\left(\frac{\partial g}{\partial d}\right) = \left(\frac{\partial g}{\partial d}\right)^* + \left(\frac{\partial g}{\partial m}\right)\left(\frac{\partial m}{\partial d}\right) + \left(\frac{\partial g}{\partial u}\right)\left(\frac{\partial u}{\partial d}\right), \qquad \left(\frac{\partial d}{\partial S}\right) = \frac{2d}{S}, \qquad \left(\frac{\partial d}{\partial Z}\right) = \frac{-2d}{Z},$$
$$\left(\frac{\partial u}{\partial s}\right) = \sqrt{\frac{(a-d)(b-c)}{ad}}, \qquad \left(\frac{\partial L}{\partial Z}\right)^* = \left(\sqrt{\frac{a-d}{a-c}}\right)\mathrm{dn}(u|m), \qquad \left(\frac{\partial L}{\partial d}\right)^* = \frac{-L}{2(a-d)},$$
$$\left(\frac{\partial L}{\partial m}\right) = \frac{1}{2}\left(\sqrt{\frac{a-d}{a-c}}\right)Z\mathrm{sn}(u|m)\left[\mathrm{cn}(u|m)\left\{u - \mathrm{pn}(u,m|m)\right\} - \left(\frac{\mathrm{sn}(u|m)}{\mathrm{dn}(u|m)}\right)\right],$$
$$\left(\frac{\partial L}{\partial u}\right) = -\left(m\sqrt{\frac{a-d}{a-c}}\right)Z\mathrm{sn}(u|m)\mathrm{cn}(u|m), \qquad \left(\frac{\partial \ell}{\partial m}\right) = \frac{\left(\sqrt{1+f}\right)\mathrm{dn}(u|m)\left[u - \mathrm{pn}(u,m|m)\right]}{2m\left[1 + f\mathrm{sn}^2(u|m)\right]},$$

$$\left(\frac{\partial \ell}{\partial u}\right) = \frac{-\left(\sqrt{1+f}\right)\mathrm{dn}(u|m)}{1+f\mathrm{sn}^2(u|m)}, \qquad \left(\frac{\partial g}{\partial d}\right)^* = \frac{c+(a-c)\mathrm{pn}(u,-f|m)}{2(a-d)\sqrt{(a-d)(b-c)}},$$

$$\left(\frac{\partial g}{\partial m}\right) = \left(\frac{a-c}{2\sqrt{(a-d)(b-c)}}\right)\left[u + \frac{\mathrm{pn}(u,m|m) - \mathrm{pn}(u,-f|m)}{m+f}\right],$$

$$\left(\frac{\partial g}{\partial u}\right) = \frac{1}{\sqrt{(a-d)(b-c)}}\left[c + \frac{a-c}{1+f\mathrm{sn}^2(u|m)}\right], \qquad \left(\frac{\partial m}{\partial d}\right) = \frac{(a-b)(a-c)}{(a-d)^2(b-c)},$$

$$\left(\frac{\partial u}{\partial d}\right) = -\left(\frac{1}{2d}\sqrt{\frac{a(b-c)}{d(a-d)}}\right)s. \tag{14}$$

5. CONCLUSION

By utilizing a new generating function, we obtain a new set of canonical variables of rotation from the Andoyer canonical variables. The new set is superior to the existing sets of canonical variables of rotation in the sense that (1) the unperturbed Hamiltonian remains finite while it vanishes in the case of Serret's canonical elements (Serret 1866), (2) the new variables are the rotational elements for any triaxial rigid body while the Andoyer canonical variables (Andoyer 1923) become elements only when the body is oblately spheroidal and while their 1-2-3 variations (Fukushima 1994) do so only when the body is prolately spheroidal, (3) the new set is universal while Sadov's canonical elements (Sadov 1970) as well as Kinoshita's similar elements (Kinoshita 1972) are well-defined only for the short-axis mode rotation, and (4) the transformation to the Andoyer variables is explicit while it is implicit for Sadov's and Kinoshita's canonical elements. In short, the new canonical elements are a universal counterpart, in the rotational dynamics, of the Delaunay canonical elements for elliptic orbital motion.

6. REFERENCES

Andoyer, H. 1923, Mécanique Céleste, Gauthier-Villars, Paris

Byrd, P.F., & Friedman, M.D. 1971, Handbook on Elliptic Integrals for Engineers and Physicists, 2nd ed. (Berlin: Springer-Verlag)

Fukushima, T. 1994, Celest. Mech. Dyn. Astron., 60, 57.

Fukushima, T. 2008a, AJ , 135, 72.

Fukushima, T. 2008b, AJ , 135, 2298.

Fukushima, T. 2008c, AJ , 136, 649.

Fukushima, T. 2008d, AJ , 136, 1728.

Goldstein, H., Poole, C.P., & Safko, J.L. 2002, Classical Mechanics, 3rd ed., (Reading, MA: Addison-Wesley)

Gurfil, P., Elipe, A., Tangren, W., & Efroimsky, M. 2007, Regular and Chaotic Dynamics, 12, 389.

Hughes, P.C. 1986, Spacecraft Attitude Dynamics, (New York: John Wiley & Sons)

Kinoshita, H. 1972, Publ. Astron. Soc. Japan, 24, 423.

MacMillan, W.D. 1936, Dynamics of Rigid Bodies, reprinted 1960 (New York: Dover)

Sadov, Y.A. 1970, Keldysh Inst. Appl. Math. Acad. Sci. USSR, Preprint No.22

Serret, J.A. 1866, Mem. Acad. Sci. Paris, 35, 585.

Wolfram, S. 2003, The Mathematica Book, 5th ed., (Cambridge: Wolfram Research Inc./Cambridge Univ. Press)