ANALYTICAL COMPUTATION OF THE TRANSLATIONAL INTERNAL MOTION OF A SIMPLE NON-ISOBARYCENTRIC THREE-LAYER EARTH MODEL

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ABSTRACT. In this investigation, we start the study of the dynamics of non-isobarycentric three-layer celestial bodies, that is to say, of bodies whose internal layers have barycenters located in different points of the space. Due to the complexity of the problem, our first step focuses on determining the translational internal motions of a simple three-layer body composed by a spherical rigid mantle, a homogeneous fluid outer core and a spherical rigid inner core. To this end, we construct the Lagrangian of the system by considering an irrotational fluid motion. On the basis of assuming a small relative motion of the mantle with respect to the inner core, the analytical expression of its associated free frequency, usually referred as Slichter mode, is found. This expression is in agreement with others provided by quite different approaches. Finally, we discuss the dependence of the frequency with the rheological parameters that characterize the Earth model.

1. INTRODUCTION

Usually, Earth rotation studies assumes isobarycentric models, that is to say, models for which the barycenters of all the components, or layers, are located always at the same point. This is an approximation, since the rigid motions around the barycenter of the Earth also contain a translational part. In this way, we face the problem to consider non-isobarycentric bodies, in such a way that the rigid internal motions around the barycenter of the body is composed of rotational and translational motions. This is the case, for example, of a three layer Earth model composed of a solid mantle, a fluid core and a solid inner core that we will study in this investigation.

One possibility to tackle this problem is to apply the normal modes techniques (Rogister 2003). However, they present some disadvantages. For example: (1) they provide numerical solutions, what makes difficult to interpret the interdependence between the different internal motions and to analyze the influence of the Earth model characteristics in them; (2) a precise data set describing the rheology of the body is need, this information is not available for the main part of the celestial bodies. Therefore, it becomes evident a need to develop complementary analytical treatments to these kind of approaches.

As far as we know, currently there is not any analytical approach that considers the whole internal dynamics of a non-isobarycentric three–layer Earth model. The only exceptions are referred, on the one hand, to Earth rotation studies that do not consider the translation among the barycenter of the layers. On the other hand, there have been developed two investigations, due to Busse (1974) and Grinfeld & Wisdom (2005), which worked out the translational motions by constructing the linear momentum equations of the inner core with the help of vectorial mechanics by different Earth models. Busse (1974) considers an Earth model composed of a motionless spherical rigid mantle, an homogeneous fluid outer core and a spherical rigid inner core. Grinfeld & Wisdom (2005) discussed the translational internal motion of an Earth model composed of a spherical rigid mantle, an homogeneous fluid outer core and a spherical rigid inner core, neglecting the effect of the rotation. These translational motions have geophysical interest since it has been proposed (Slichter, 1961) that the internal translational motion of the inner core barycenter could be observed in precise gravimeter records after great earthquakes, providing information about the rheology in the fluid–inner core boundary. In fact, this is a simplified model since one should consider the effects of the Earth rotation.

In this note, we start the study of the dynamics of the internal motion by means of Analytical Mechanics methods, since these approaches have provided competitive Earth rotation models (e.g. Getino & Ferrándiz, 2001). The first step in this task is to give a proper description of the internal translational motion in this framework.

2. LAGRANGIAN OF THE SYSTEM

Next, we will consider a three-layer Earth model similar to that of Grinfeld & Wisdom (2005), that is to say, a body isolated in the space and composed of an external rigid spherical shell (mantle) and an internal rigid sphere (inner core). The space between these layers is filled with a perfect fluid (fluid outer core); all the components of the system are assumed to have constant densities distributions (see figure 1). In addition, we will only consider the translational motion of the rigid layers. To describe the dynamics of the system, we will construct the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,\ i = 1, \ldots, n, \tag{1}$$

being $L = T - V$ the difference between the kinetic and potential energy. The dynamical system is properly described by considering six generalized coordinates: three, $\vec{\xi}_M$, are the components of the mantle barycenter O_M with respect to a reference frame whose origin is located at barycenter of the Earth, O, and the remaining ones, $\vec{\xi}_S$, the components of the inner core barycenter O_S referred to the same frame. With respect to the motion of the fluid, let us point out that under the above mentioned conditions it is generated by that of the solids layers (Lamb, 1932). In these circumstances, the fluid velocity $\vec{v}_F$ is irrotational and we can introduce a velocity potential f , $\vec{v}_F = \vec{\nabla} f$, such as

$$\vec{\nabla}^2 f = 0 \tag{2}$$

with the boundary conditions

$$\vec{v}_F \cdot \vec{n} = \vec{V}_M \cdot \vec{n} \text{ at } \partial D_M,\ \vec{v}_F \cdot \vec{n} = \vec{V}_S \cdot \vec{n} \text{ at } \partial D_S, \tag{3}$$

being $\vec{V}_M = d\vec{\xi}_M/dt$ and $\vec{V}_S = d\vec{\xi}_S/dt$, and $\vec{n}$ the normal vector on the boundaries ∂D_M and ∂D_S (see figure 2). Anyway, it is possible a further reduction in the number of the degrees of freedom of our system. Namely, we have the two relationships

$$m_M\vec{\xi}_M + m_F\vec{\xi}_F + m_S\vec{\xi}_S = \vec{0};\ m_F\vec{\xi}_F = m_F\vec{\xi}_M + \frac{\rho_F}{\rho_S} m_S\vec{\xi}_M - \frac{\rho_F}{\rho_S} m_S\vec{\xi}_S, \tag{4}$$

being m_i the mass of each layer, m the mass of the Earth and $\vec{\xi}_F$ the components of the fluid barycenter with respect to $\mathcal{R}$. The first one reflects the fact that the barycenters of the layers are referred to O; the second one is due to the simple geometry of our model (see figure 3). Hence, our model can be described with three generalized coordinates. The most expedient choice is to take $\vec{\eta}_S = \vec{\xi}_S - \vec{\xi}_M$. In this way, the mantle and the inner core barycenters can be written as

$$\vec{\xi}_M = \frac{m_S}{m}\left(\frac{\rho_F}{\rho_S} - 1\right)\vec{\eta}_S = \alpha_M\vec{\eta}_S,\ \vec{\xi}_S = (1 + \alpha_M)\,\vec{\eta}_S. \tag{5}$$

To construct the Lagrange equations, we will assume that the instantaneous configuration (figure 2) remains always close enough to the reference configuration (figure 1). It will allow a simpler mathematical treatment of the problem, since all the functions which involves the fluid motion will be developed up to the first order in the perturbation-like parameter $e = \left|\vec{\xi}_M - \vec{\xi}_S\right| / R_S$.

The kinetic energy of the system is $T = T_M + T_S + T_F$. Since the mantle and the inner core are rigid bodies, the expression of their kinetic energy is straightforward and with the help of 5 can be written as

$$T_M + T_S = \frac{1}{2}\left[m_M\alpha_M^2 + m_S\,(1 + \alpha_M)^2\right]\left(\frac{d\vec{\eta}_S}{dt}\right)^2. \tag{6}$$

The kinetic energy of the fluid is given by

$$T_F = \frac{1}{2}\int_{fluid} \vec{v}_F^2 \rho_F\, dV = \frac{1}{2}\int_{fluid}\left(\vec{\nabla} f\right)^2 \rho_F\, dV. \tag{7}$$

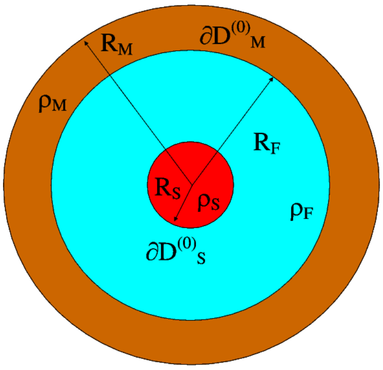

Figure 1

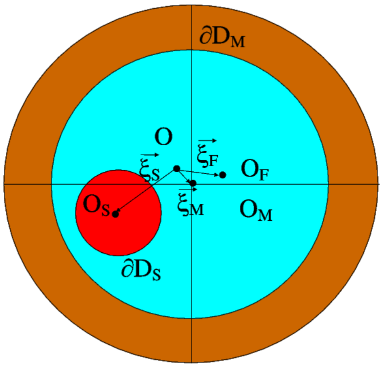

Figure 2

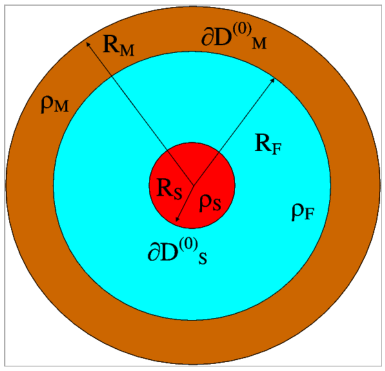

Figure 3

By considering the solution of equations 2 and 3, expressed in terms of an spherical harmonics expansion, and Gauss theorem, it can be shown (Escapa & Fukushima, 2009) that, at the first order in e, this expression has the value

$$T_F = \frac{1}{2}\left(\frac{2}{3}\pi\rho_F c\right)\left(\frac{d\vec{\eta}_S}{dt}\right)^2 = \frac{1}{2}\left\{\frac{2}{3}\pi\rho_F\left[2\alpha_M^2 R_F^3 - 2\left(1+\alpha_M\right)^2 R_S^3 + 3\frac{R_F^3 R_S^3}{R_F^3 - R_S^3}\right]\right\}\frac{d\vec{\eta}_S}{dt}^2. \tag{8}$$

The potential energy of the system is due to the gravitational interaction among the layers. However, since the mantle is an rigid spherical shell of constant density the gravitational potential inside it takes a constant value, so it does not play any role in the dynamics of the system. Hence, the gravitational potential energy of our model arises from the fluid–solid inner core interaction. Its expression can be computed by considering the the potential gravitational energy of a homogeneous sphere and the decomposition sketched in the figure 3. Namely, we have

$$V = \frac{2\pi}{3}G\rho_F\, m_S\left(1-\frac{\rho_F}{\rho_S}\right)\left(\vec{\eta}_S\right)^2. \tag{9}$$

Taking into account the equations 6, 8 and 9, the Lagrangian of the system turns out to be

$$L = \frac{1}{2}\left[m_M\alpha_M^2 + m_S\left(1+\alpha_M\right)^M + \frac{3}{2}\pi\rho_F c\right]\left(\frac{d\vec{\eta}_S}{dt}\right)^2 - \frac{1}{2}\left[\frac{4\pi}{3}G\rho_F\, m_S\left(1-\frac{\rho_F}{\rho_S}\right)\right]\left(\vec{\eta}_S\right)^2. \tag{10}$$

3. DISCUSSION

Equation 10 shows that the dynamics of the system is completely specified since it is the same as that of a harmonic oscillator of frequency. Therefore, the barycenter of the solid inner core performs harmonics oscillations with respect to the mantle barycenter, being the equilibrium position the reference configuration. The dependence of this translational internal motion with the characteristics of the Earth model is clearly reflected in the functional dependence of the frequency, ω_0, which with the aid of the equations 5 and 8 can be written as

$$\omega_0^2 = \frac{4\pi}{3}G\rho_F\left(1-\frac{\rho_F}{\rho_S}\right)\frac{1}{\left[1+\frac{1}{2}\frac{\rho_F}{\rho_S}+\frac{3}{2}\left(\frac{\rho_F}{\rho_S}\right)^2\frac{m_S}{m_F}\right]-\left[\frac{m_S}{m_M+m_F+m_S}\left(1-\frac{\rho_F}{\rho_S}\right)^2\right]}. \tag{11}$$

This expression allows easy comparisons with the previous investigations by Busse (1974) and Grinfeld & Wisdom (2005). In particular, we can recover the results of Busse by equating the rotation to 0 in his formula, and by taking $m \to +\infty$ (mantle rest condition) in equation 11. This is also the case for the expression of Slichter mode of Grinfeld & Wisdom (2005) which, after some tedious algebra, agrees with equation 11 in spite of the quite different methods employed.

On the other hand, the derived expression of the Slichter frequency ω_0 makes possible to discuss the influence of the model on it. In particular, we can compute for different three-layer celestial bodies the effect that has the motion of the mantle in the frequency. The contribution of this effect appears in the denominator of equation 11 through the term

$$\Delta = \frac{m_S}{m_M + m_F + m_S} \left(1 - \frac{\rho_F}{\rho_S}\right)^2 . \tag{12}$$

By considering some numerical data given in Grinfeld & Wisdom (2005) for the Earth and Mercury, we can compute the period T_{ω_0} associated to the frequency ω_0; its counterpart when considering a mantle in rest $T_{\omega_0|_{\vec{\xi}_M=\vec{0}}}$ and the absolute value of the relative difference between these two periods. The results are displayed in the next table.

Body	T_{ω_0}	$T_{\omega_0\vert_{\vec{\xi}_M=\vec{0}}}$	Rel. dif.%	m_S/m	$1-\rho_F/\rho_S$
Earth	4.2356	4.2358	.003	.016	.077
Mercury	8.3900	8.3977	.091	.598	.158

As it can be seen, the contribution of the mantle motion to the period in the case of the Earth is completely negligible. This is due to two factors. On the one hand, to the small mass of the inner core with respect to the mass of the whole planet. On the other one, the relative density contrast between the fluid an the inner core, $1 - \rho_F/\rho_S$, is also small. So, as Busse (1974) did rightly the effect of the mantle motion in the case of the Earth can be safely disregarded when computing the value of the Slichter mode. In addition, following Grinfeld & Wisdom (2005), we can examine the Slichter mode of Mercury. For this planet the mass of the inner core is more than the half of the total mass, so one could expect the effect of the mantle motion to be significant (Grinfeld & Wisdom, 2005). However, that is not the case. With help of equation 12 derived following our analytical treatment, the reason is clear since also for Mercury the relative density contrast between the fluid an the inner core is very small, so when taking its square in equation 12 the value of Δ still remains small.

To summarize, it has been developed a model to describe Earth translational internal motions with the methods of Analytical Mechanics, providing an analytical expression of Slichter mode. In particular, it have been shown that the effect of the mantle motion in the value of the Slichter mode should be considered depending both on the existence of a heavy inner core and the value of the density contrast, conditions that are not fulfilled by the planets Earth and Mercury but that could be verified in the case of other celestial bodies. Future steps of this investigation should consider the joint effects of the rotational and translational internal motions, as well as improvements in the rheological characterization of the Earth model.

Acknowledgements. AE's contribution has been partially supported by the Descartes–Nutation project and the Spanish projects I+D+I, AYA2004-07970 and AYA2007-67546 and *Junta de Castilla y León* project VA070A07.

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