COMPATIBILITY OF THE IERS EARTH ROTATION REPRESENTATION AND ITS RELATION TO THE NRO CONDITIONS

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The rotation of the earth is mathematically represented by an orthogonal rotation matrix $\mathbf{R}(t)$, which relates coordinates \mathbf{x}_C in a Celestial Reference System $\vec{\mathbf{e}}^C$ with coordinates \mathbf{x}_T in a Terrestrial Reference System $\vec{\mathbf{e}}^T$ according to

$$\mathbf{x}_C = \mathbf{R}\mathbf{x}_T, \qquad \vec{\mathbf{e}}^T = \vec{\mathbf{e}}^C \mathbf{R}, \tag{1}$$

where any row-matrix $\vec{\mathbf{e}} = [\vec{e_1} \ \vec{e_2} \ \vec{e_3}]$ contains the elements $\vec{e_1}, \vec{e_2}, \vec{e_3}$ of a corresponding geocentric orthonormal basis and $\mathbf{x} = \begin{bmatrix} x^1 \ x^2 \ x^3 \end{bmatrix}^T$ is the column-matrix of the relevant coordinates. The orthogonal matrix $\mathbf{R}(t)$ can be adequately defined by 3 parameters (functions of time), e.g. Euler angles. Consequently any representation involving k parameters must be accompanied by k-3conditions between these parameters. Once $\mathbf{R}(t)$ is given by a particular representation, the rotation of the earth (in fact of the chosen terrestrial reference system) is completely defined and so is the corresponding earth rotation vector (vector of instantaneous angular rotation) $\vec{\omega} = \vec{\mathbf{e}}^T \boldsymbol{\omega}_T = \vec{\mathbf{e}}^C \boldsymbol{\omega}_C$. In fact $\vec{\omega}$, is defined by its terrestrial coordinates $\boldsymbol{\omega}_T$, which can be derived from \mathbf{R} by the generalized Euler's kinematic equations

$$[\boldsymbol{\omega}_T \times] = \mathbf{R}^T \frac{d\mathbf{R}}{dt} \,, \tag{2}$$

where $[\boldsymbol{\omega}_T \times]$ denotes the antisymmetric matrix having $\boldsymbol{\omega}_T$ as its axial vector. Both the classical astronomical representation and the new one provided by IERS in accordance with the IAU 2000 resolutions involve a separation of **R** into 3 parts $\mathbf{R} = \mathbf{Q}\mathbf{D}\mathbf{W}$. The precession-nutation part \mathbf{Q} , the diurnal rotation part \mathbf{D} and the polar motion part \mathbf{W} , are realized with the use of two intermediate systems, the Intermediate Celestial (IC) system $\vec{\mathbf{e}}^{IC} = \vec{\mathbf{e}}^C \mathbf{Q}$ and the Intermediate Terrestrial (IT) system $\vec{\mathbf{e}}^{IT} = \vec{\mathbf{e}}^C \mathbf{Q}\mathbf{D} = \vec{\mathbf{e}}^T \mathbf{W}^T$ with a common third axis, defined by the unit vector

$$\vec{p} = \vec{e}_3^{IC} = \vec{e}_3^{IT} \,, \tag{3}$$

which we will loosely call the "celestial pole". In fact the diurnal part is simply a rotation around the celestial pole \vec{p}

$$\mathbf{D} = \mathbf{R}_3(-\theta) \,. \tag{4}$$

The question arises about the coincidence or not of the celestial pole (CP) \vec{p} used in a specific representation with the (mathematically) "compatible celestial pole" (CCP) defined through (2) and represented by the unit vector

$$\vec{n} = \frac{1}{|\vec{\omega}|} \,\vec{\omega} \,. \tag{5}$$

In addition to the directional coincidence, the question arises about the magnitude coincidence between the compatible angular velocity $\omega = |\vec{\omega}|$ and the angular velocity of the diurnal rotation $\dot{\theta} = d\theta/dt$.

A three-part earth rotation representation involves at least 5 parameters according to the scheme (Dermanis, 1977) $\mathbf{R} = \mathbf{Q}(\Xi, \mathbf{H})\mathbf{D}(\psi)\mathbf{W}(\xi, \eta)$, which however suffers from the fact that the separation between diurnal rotation on one part and precession-nutation or polar motion on the other, strongly depends on the explicit parameterizations used. For example the three possible choices $\mathbf{Q} = \mathbf{R}_1(\Xi)\mathbf{R}_2(\mathbf{H})$, $\mathbf{Q} = \mathbf{R}_2(B)\mathbf{R}_1(A)$, $\mathbf{Q} = \mathbf{R}_3(-E)\mathbf{R}_2(-d)\mathbf{R}_3(E)$, lead to a different position of the "celestial origin" \vec{e}_1^{IC} (starting direction for the "diurnal" angle ψ), while the same is true for the "terrestrial origin" \vec{e}_1^{IT} (terminal direction for ψ) with respect to the possible choices $\mathbf{W} = \mathbf{R}_1(-C)\mathbf{R}_2(-D)$, $\mathbf{W} = \mathbf{R}_2(-\Pi)\mathbf{R}_1(-\Sigma)$, $\mathbf{W} = \mathbf{R}_3(-F)\mathbf{R}_2(g)\mathbf{R}_3(F)$. To avoid this problem the IAU2000 resolutions introduce a separation $\psi = s + \theta - s'$, including s in precession-nutation \mathbf{Q} and s' in polar motion \mathbf{W} , in such a way that the positions of the celestial \vec{e}_1^{IC} and terrestrial origin \vec{e}_1^{IT} are independently defined with the ingenious introduction of the 2 Non Rotating Origin (NRO) conditions.

More details can be found in the IERS conventions 2003 (McCarthy and Petit, 2004), as well as in a series of relevant papers (Capitaine, N., 1986, Capitaine & Gontier, 1993, Capitaine et al. 2000, 2002, IERS, 2001, Johnston et al., 2000, Lambert & Bizouard, 2002, McCarthy, 1996, Seidelmann & Kovalevsky, 2002). The specific IERS representation has the form

$$\mathbf{R} = \mathbf{Q}(E, d, s)\mathbf{D}(\theta)\mathbf{W}(F, g, s')$$
$$= \mathbf{R}_3(-E)\mathbf{R}_2(-d)\mathbf{R}_3(E)\mathbf{R}_3(s)\mathbf{R}_3(-\theta)\mathbf{R}_3(-s')\mathbf{R}_3(-F)\mathbf{R}_2(g)\mathbf{R}_3(F)$$
(6)

where the appearing 7 parameters are in fact reduced to 5, by means of the NRO conditions $\dot{s} = \dot{E}(\cos d - 1), \dot{s}' = \dot{F}(\cos g - 1)$ (dots denote derivatives with respect to time), which give the dependence relations s = s(E, d), s' = s'(F, g).

Generalizing and extending previous work (Dermanis, 2003), we will examine here the possibility of reducing the 7 parameters of the earth rotation representation (6) to the minimum required of 3 independent parameters, by introducing 4 appropriate conditions. Three candidate conditions are the 2 *direction conditions* implied by

$$\vec{p} = \vec{n}$$
, $\left(\vec{e}_3^{IC} \equiv \vec{e}_3^{IT} = \frac{1}{|\vec{\omega}|} \vec{\omega}\right)$ (7)

and the magnitude condition

$$\omega \equiv |\vec{\omega}| = \dot{\theta} \equiv \frac{d\theta}{dt} \,. \tag{8}$$

The one condition which is still missing can be associated with the choice of the origin \vec{e}_1^{IC} of the diurnal rotation angle θ . Note that once this choice is made, the value $\theta = \int \omega dt$ resulting from (8) defines uniquely the terminating direction \vec{e}_1^{IT} . For any choice of E, d, s, θ , s', F, g, replacement of s and s' with $\bar{s} = s + \Delta s$ and $\bar{s}' = s' - \Delta s$, respectively, provides exactly the same rotation matrix **R**. Thus the 4th condition is implicitly provided by the arbitrary choice of the function $\Delta s(t)$, with every choice corresponding to a different choice of \vec{e}_1^{IC} (and consequently of \vec{e}_1^{IT}).

In order to find 4 explicit conditions we need to resort to the NRO conditions, which we will present in a geometric form, by using the decomposition

$$\vec{\omega} = \vec{\omega}_Q + \vec{\omega}_D + \vec{\omega}_W \,. \tag{9}$$

The relative rotation vectors $\vec{\omega}_Q = \vec{\mathbf{e}}^{IC}(\omega_Q)_{IC}$ of $\vec{\mathbf{e}}^{IC}$ with respect to $\vec{\mathbf{e}}^C$, $\vec{\omega}_D = \vec{\mathbf{e}}^{IT}(\omega_D)_{IT}$ of $\vec{\mathbf{e}}^{IT}$ with respect to $\vec{\mathbf{e}}^{IC}$ and $\vec{\omega}_W = \vec{\mathbf{e}}^T(\omega_W)_T$ of $\vec{\mathbf{e}}^T$ with respect to $\vec{\mathbf{e}}^{IT}$, are uniquely defined by their components determined respectively from relations similar to (2)

$$\left[(\boldsymbol{\omega}_Q)_{IC} \times \right] = \mathbf{Q}^T \frac{d\mathbf{Q}}{dt} , \quad \left[(\boldsymbol{\omega}_D)_{IT} \times \right] = \mathbf{D}^T \frac{d\mathbf{D}}{dt} , \quad \left[(\boldsymbol{\omega}_W)_T \times \right] = \mathbf{W}^T \frac{d\mathbf{W}}{dt} . \tag{10}$$

The two NRO conditions take the geometric form $\vec{\omega}_Q \perp \vec{p}, \, \vec{\omega}_W \perp \vec{p}$ or

$$\vec{\omega}_Q \bullet \vec{p} = 0, \qquad \vec{\omega}_W \bullet \vec{p} = 0, \tag{11}$$

which impose "no component" along the diurnal rotation direction \vec{p} , for both the precessionnutation and polar motion relative rotation vectors $\vec{\omega}_Q$ and $\vec{\omega}_W$, respectively. Now we have altogether 5 conditions, the 2 direction conditions (7), the magnitude condition (8) and the 2 NRO conditions (11), among which we must isolate the desired 4 independent conditions. To do this we need to switch from the geometric to a mathematical form by employing components in the most advantageous intermediate celestial system $\vec{\mathbf{e}}^{IC}$. Since $\vec{p} = \vec{\mathbf{e}}^{IC} \mathbf{p}_{IC} \equiv \vec{e}_3^{IC}$ has components $\mathbf{p}_{IC} = [0 \ 0 \ 1]^T \equiv \mathbf{i}_3$ and $\vec{n} = \vec{\mathbf{e}}^{IC} \mathbf{n}_{IC} = \omega^{-1}\vec{\omega}$, the direction conditions $\vec{n} = \vec{p}$ take the form $n_{IC}^1 = p_{IC}^1 = 0$, $n_{IC}^2 = p_{IC}^2 = 0$, or in terms of the rotation vector components

$$\omega_{IC}^1 = 0 , \qquad \omega_{IC}^2 = 0 . \tag{12}$$

Setting $\vec{\omega}_Q = \vec{\mathbf{e}}^{IC}(\boldsymbol{\omega}_Q)_{IC}, \, \vec{\omega}_W = \vec{\mathbf{e}}^{IC}(\boldsymbol{\omega}_W)_{IC}$ the NRO conditions become accordingly

$$(\omega_Q)_{IC}^3 = 0, \qquad (\omega_W)_{IC}^3 = 0.$$
 (13)

For the remaining magnitude condition $\omega \equiv |\vec{\omega}| = \sqrt{\omega_{IC}^T \omega_{IC}} = \dot{\theta}$ we have according to (9) $\omega_{IC} = (\omega_Q)_{IC} + (\omega_D)_{IC} + (\omega_W)_{IC}$, where , $(\omega_D)_{IC} = (\omega_D)_{IT} = \dot{\theta}\mathbf{i}_3$ and $\omega_{IC}^3 = (\omega_Q)_{IC}^3 + (\omega_D)_{IC}^3 + (\omega_W)_{IC}^3$, where $(\omega_D)_{IC}^3 = \dot{\theta}$. Thus the magnitude condition in the form $\omega^2 = \omega_{IC}^T \omega_{IC} = \dot{\theta}^2$, becomes

$$\omega^{2} = [\omega_{IC}^{1}]^{2} + [\omega_{IC}^{2}]^{2} + [\omega_{IC}^{3}]^{2} =$$

= $[\omega_{IC}^{1}]^{2} + [\omega_{IC}^{2}]^{2} + [(\omega_{Q})_{IC}^{3} + \dot{\theta}^{2} + (\omega_{W})_{IC}^{3}]^{2} = \dot{\theta}^{2}.$ (14)

From the above relation it is obvious that when the direction conditions $\omega_{IC}^1 = 0$, $\omega_{IC}^2 = 0$ and the NRO conditions $(\omega_Q)_{IC}^3 = 0$, $(\omega_W)_{IC}^3 = 0$ are satisfied, then the magnitude condition (14) is also trivially satisfied. Therefore we arrive at the following conclusion:

The 2 direction conditions (12) and the 2 NRO conditions (13), provide the desired set of 4 independent conditions, which reduce the 7 parameters E, d, s, θ , s', F, g, to the required 3 independent parameters that suffice for the description of the orthogonal rotation matrix \mathbf{R} .

The current IERS representation implements the NRO conditions but not the direction conditions and consequently $\vec{n} \neq \vec{p}$ and $\omega \neq \dot{\theta}$. Thus the compatible celestial pole (CCP) \vec{n} does not coincide with the celestial pole \vec{p} , which in this case is the Celestial Intermediate Pole (CIP). The CIP results by removing from the theoretical solution to the precession-nutation problem, high frequency terms of (celestial) periods smaller than 2 days. Unlike precession-nutation, polar motion and length-of-the-day variation cannot be predicted by theory and therefore the rotation matrix **R** is constructed from observations, which have a resolution limited by the corresponding sampling rate. The removal of high frequency terms from the formerly used Celestial Ephemeris Pole (CEP), has thus brought the CIP closer to the CCP but a difference still remains. Apart from the direction problem, the compatible angular velocity ω provides the means for defining a compatible diurnal rotation angle $\theta_{CCP} = \int \omega dt$, which will in turn defines a compatible Universal Time UT_{CCP} = $A\theta_{CCP} + B$, using appropriate constants Aand B in order to conveniently choose the origin and unit of this new time system. The IERS provided parameters can be used for determining the direction of the corresponding CCP and compatible angular velocity ω . Setting $\mathbf{n}_{C} = [X_{CCP}Y_{CCP}Z_{CCP}]^{T}$ for the celestial components, while $\mathbf{p}_C = [X Y Z]^T$ and $\mathbf{p}_T = [\xi \eta \zeta]^T$, then with X, Y, $\xi \approx x_P$, $\eta \approx -y_P$ and θ provided by the IERS, we may compute first $\boldsymbol{\omega}_C = \mathbf{R}\boldsymbol{\omega}_T$, with $\boldsymbol{\omega}_T$ from (2) and next $\boldsymbol{\omega} = (\boldsymbol{\omega}_C^T \boldsymbol{\omega}_C)^{1/2}$ and $\mathbf{n}_C = \boldsymbol{\omega}^{-1} \boldsymbol{\omega}^C$. After rather lengthy computations, which also implement the NRO conditions, we arrive at

$$\omega^{2} = \dot{\theta}^{2} + \left(b_{Q}^{1}\right)^{2} + \left(b_{Q}^{2}\right)^{2} + \left(b_{W}^{1}\right)^{2} + \left(b_{W}^{2}\right)^{2} - 2\cos\psi\left(b_{Q}^{1}b_{W}^{1} + b_{Q}^{2}b_{W}^{2}\right) + 2\sin\psi\left(b_{Q}^{1}b_{W}^{2} - b_{Q}^{2}b_{W}^{1}\right) \quad (15)$$

$$\begin{bmatrix} X_{CCP} \\ Y_{CCP} \end{bmatrix} = \frac{\dot{\theta}}{\omega} \begin{bmatrix} X \\ Y \end{bmatrix} + \frac{1}{\omega} \begin{bmatrix} 1 - \alpha X^{2} & \alpha XY \\ -\alpha XY & 1 - \alpha Y^{2} \end{bmatrix} \left\{ \begin{bmatrix} b_{Q}^{1} \\ b_{Q}^{2} \end{bmatrix} - \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} b_{W}^{1} \\ b_{W}^{2} \end{bmatrix} \right\} \quad (16)$$

where $\psi = -s + \theta + \dot{s}$ and

$$\begin{bmatrix} b_Q^1 \\ b_Q^2 \end{bmatrix} = \frac{Y\dot{X} - X\dot{Y}}{X^2 + Y^2} \begin{bmatrix} X \\ Y \end{bmatrix} + \frac{X\dot{X} + Y\dot{Y}}{(X^2 + Y^2)\sqrt{1 - X^2 - Y^2}} \begin{bmatrix} -Y \\ X \end{bmatrix},$$
(17)

$$\begin{bmatrix} b_W^1 \\ b_W^2 \end{bmatrix} = \frac{\eta \dot{\xi} - \xi \dot{\eta}}{\xi^2 + \eta^2} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \frac{\xi \dot{\xi} + \eta \dot{\eta}}{(\xi^2 + \eta^2)\sqrt{1 - \xi^2 - \eta^2}} \begin{bmatrix} -\eta \\ \xi \end{bmatrix}.$$
 (18)

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