EFFICIENT INTEGRATION OF TORQUE-FREE ROTATION BY ENERGY SCALING METHOD

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ABSTRACT. As a first trial of the manifold correction methods applied to rotational motions, we adapted its simplest technique, the energy scaling method, to the torque-free rotational motion in terms of Serret-Andoyer variables. The key point is to keep rigorously the consistency of the kinetic energy relation by applying a scaling to L, the C-axis component of the rotational angular momentum at every integration step. As a result, the new method suppress the growth rate of the integration errors in the combined rotational angles, $g + \ell$, from quadratic to linear in time.

1. ENERGY SCALING METHOD

Originally developed by Nacozy (1971), and extended to the restricted three body problem by Murrison (1989), the manifold correction methods have revived recently in orbital integrations. As for the list of our works on the manifold correction methods, see the references of our latest paper (Fukushima, 2005). Very recently, we extended the method to cover the case of conservative potential and succeeded in reducing the error growth of the two-body problem perturbed by J_2 and other zonal harmonics (Umetani and Fukushima, 2005).

Now, let us apply it to the torque-free rotational motion, the equation of motion of which is described by the well-known Serret-Andoyer canonical variables (L, G, H, ℓ, g, h) (Andoyer, 1923; Deprit, 1967; Kinoshita, 1972; 1977; 1992) as

$$\frac{\mathrm{d}L}{\mathrm{dt}} = -\left(\frac{a+b}{2}\right)\left(G^2 - L^2\right)\sin 2\ell, \tag{1}$$

$$\frac{\mathrm{d}\ell}{\mathrm{dt}} = -L\left[\left(\frac{a+b}{2} - c\right) - \left(\frac{a-b}{2}\right)\cos 2\ell\right], \tag{2}$$

$$\frac{\mathrm{d}g}{\mathrm{dt}} = G\left[\left(\frac{a+b}{2}\right) - \left(\frac{a-b}{2}\right)\cos 2\ell\right], \tag{3}$$

and

$$\frac{\mathrm{d}G}{\mathrm{dt}} = \frac{\mathrm{d}H}{\mathrm{dt}} = \frac{\mathrm{d}h}{\mathrm{dt}} = 0. \tag{4}$$

Here $a \equiv 1/A$, $b \equiv 1/B$, and $c \equiv 1/C$ are the reciprocals of the principal moments of inertia, A, B, and C, respectively. Note that conserved are G, H, h, and the kinetic energy,

$$T \equiv \frac{1}{4}\left[(a+b)G^2 - (a+b-2c)L^2 - (a-b)\left(G^2 - L^2\right)\cos 2\ell\right]. \tag{5}$$

In order to maintain the consistency condition of T, we modify the integrated L by the scaling $L \rightarrow sL$ at every integration step. Here the scaling factor s is determined uniquely as

$$s = \sqrt{\frac{G_0^2 \left[(a+b) - (a-b)\cos 2\ell\right] - 4T_0}{L^2 \left[(a+b-2c) - (a-b)\cos 2\ell\right]}}. \tag{6}$$

where G_0 and T_0 are their initial values while L and ℓ are the integrated values. Note that we do not correct any other integrated variables, ℓ nor g.

2. NUMERICAL EXPERIMENTS

In order to examine the effectiveness of the energy scaling method, we conducted test integrations of three typical cases; a triaxial rigid Earth, a triaxial rigid Moon, and the asteroid Ida. The rotational motion of an oblate spheroidal rigid body, where $A = B$ and therefore $a = b$, in terms of the Serret-Andoyer variables reduces to be trivial linear motions in g and ℓ and a constant L. Therefore, it cannot be used as a practical test of numerical integrations.

In the below, we will show the case of the so-callede short-axis mode, namely the case where the polhode is a curve circulating around the axis of the largest principal moment of inertia, the C-axis. In this mode, the solution in g and ℓ are circulating while L is librating (Kinoshita, 1992). Namely the first two angle-like variables, g and ℓ become quasi-linear functions of time with relatively small periodic terms while the action-like variable L is a purely periodic function with a constant offset. Then the numerical integration errors most significantly appear in the sum of two angle-like variables, $g + \ell$.

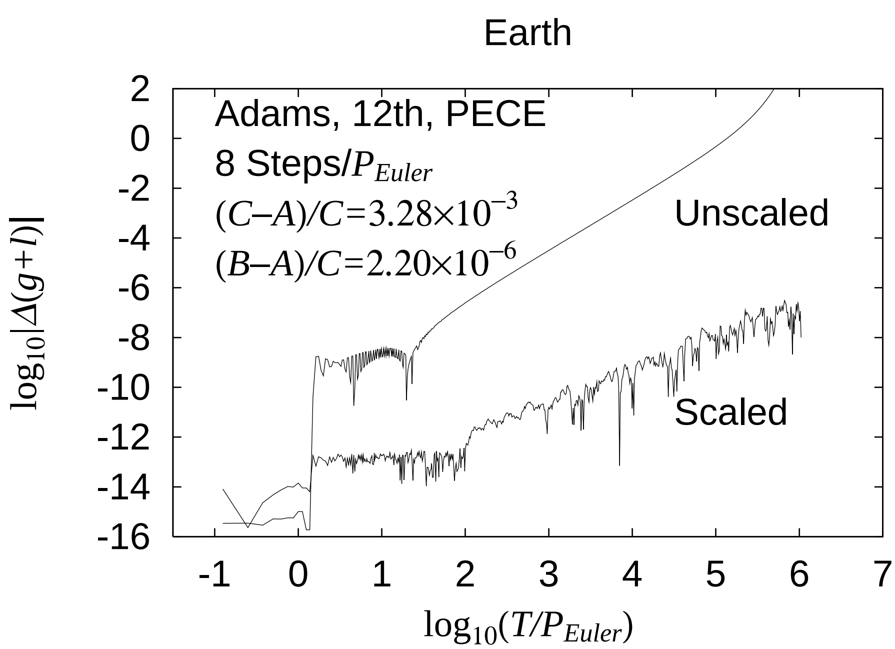

Figure 1: The numerical integration errors in the combined rotational angles $g + \ell$ of the torque-free rotation of a rigid triaxial Earth are shown as functions of time in a log-log scale. The initial angle of inclination, J_0, is chosen as small as 10^{-6} radian. The step size was as large as $1/8$ of the Eulerian period, ~ 300 days.

See Figure 1, which compares the error growth of the torque-free rotation of a rigid triaxial Earth obtained by the standard method to integrate the equation of motion directly and that with the aforementioned energy scaling with respect to the kinetic energy, T. This is the case of almost oblately spheroidal body, i.e. $B - A \ll C - A \ll C$.

In the figure, we illustrated the errors in the combined rotational angles $g + \ell$ as functions of time in a log-log scale. We compared the error growth of the torque-free rotation of a rigid triaxial Earth integrated by two methods; (1) the standard method to integrate directly the canonical equation of rotational motion in terms of Serret-Andoyer variables, and (2) the energy scaling method described in the previous section.

We adopted the 12th order implicit Adams method in the PECE mode (predict, evaluate, correct, evaluate), fixed the step size through the integration, and set it as large as 1/8 the Eulerian period. As for the starting tables needed for Adams method, we prepared them by Gragg's extrapolation method. We measured the errors by comparing with the reference solutions obtained by the same method, the same integrator, and the same model parameters but with half the step size. Since the order of the integrators are sufficiently high, halving the step size eliminates almost all the truncation errors.

We set the initial conditions as $g_0 = \ell_0 = 0$, and $L_0 = G_0 \cos J_0$ where G_0 is taken as unity after an appropriate adustment of units. Also we chose the initial value of the amplitude of polar motion, J_0, as small as 10^{-6} radian to resemble the actual Earth rotation.

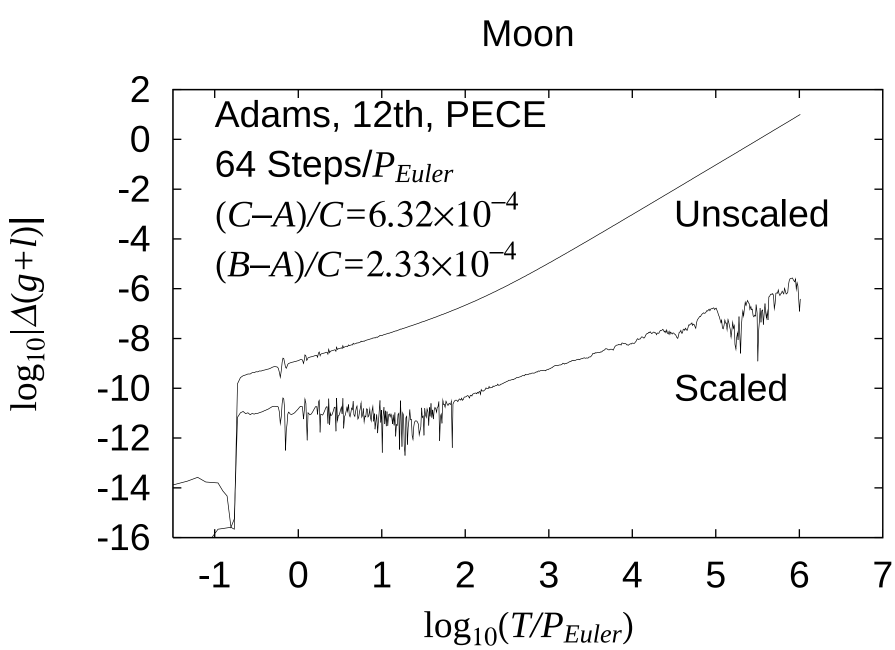

Figure 2: Same as Figure 1 but for the torque-free rotation of a rigid Moon. The initial angle J_0 was taken as 10^{-5} radian. The chosen step size was relatively small as $1/64$ of its Eulerian period.

Figure 2 shows the case of Moon. This is an example of triaxial but almost spherical case, i.e. $B - A \sim C - A \ll C$.

Figure 3 depicts the case of an illustration of truly triaxial body, i.e. $A \sim B \sim C$, namely that of the asteroid Ida.

The curves in these three graphs show that the quadratic increase of the integration errors by the standard method is reduced to a linear growth, which has been observed for a limited type of integration scheme; the symplectic integrator and the symmetric linear multistep method. The observed difference in the rate of error growth lead to a large difference in the magnitude of

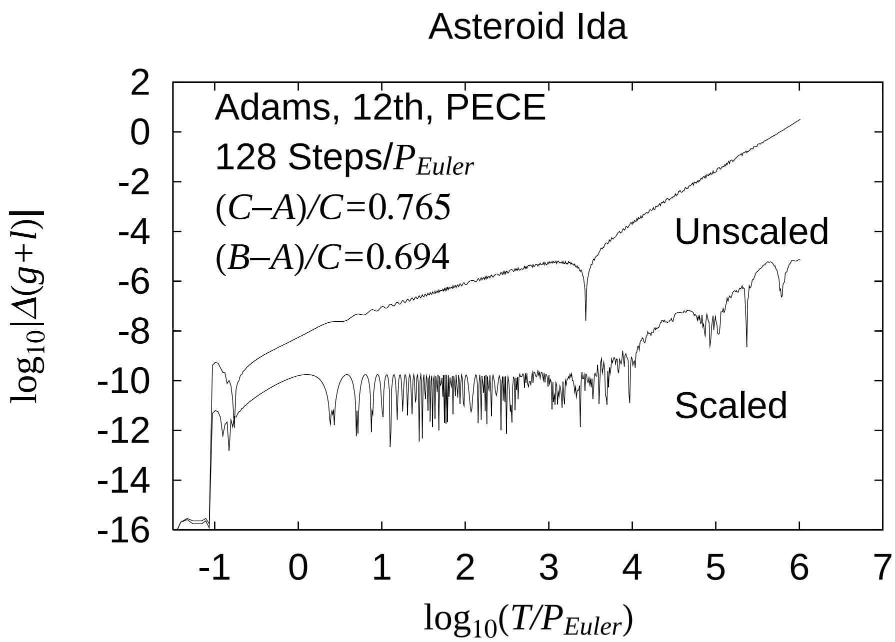

Figure 3: Same as Figure 1 but for the asteroid Ida. The initial angle J_0 was taken as relatively large 0.1 radian. The chosen step size was 1/128 of its Eulerian period.

integration error in the long run as shown in the graphs.

The next target of our study is, of course, the rotational motion under torques. In those perturbed cases, T is no longer a constant of integration. Then we should follow its time development simultaneously as we did for the Kepler energy in the first scaling method of ours (Fukushima, 2003).

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