RELATIVITY IN THE PROBLEMS OF EARTH ROTATION AND ASTRONOMICAL REFERENCE SYSTEMS: STATUS AND PROSPECTS

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ABSTRACT. Status and prospects of the problems: Earth rotation and astronomical reference systems in the framework of relativity are discussed. Several points where future work is urgently necessary are pointed out.

1. RELATIVITY IN THE PROBLEM OF EARTH'S ROTATION

The description of rotational motion of extended bodies within Einstein's theory of gravity in the general case without approximation can only be achieved by direct numerical integration of Einstein's field equations, that is by methods of numerical relativity. Even the definitions of quantities like total angular-momentum or angular velocity presents a fundamental obstacle. Only in the idealized stationary case of a single uniformly rotating axisymmetric body the scalar quantities of spin, angular velocity Ω and principle moment of inertia are well defined (e.g., Komar 1959). The extremely useful *Newtonian* concept of a rigid body that for many purposes serves as first approximation to the real problem becomes meaningless in relativity where the sound speed is limited to the speed of light. Nevertheless, in a remarkable paper Thorne and Gürsel (1983) have shown that if one restricts to first order terms in Ω rigid bodies can be introduced in general relativity and one finds a Newtonian-like Euler theory apart from the usual MacCullagh relations between the potential coefficients (multipole moments) and the components of the moment of inertia tensor I_{ab}. In mathematical terms, in general

$$M_{ab} \neq -\mathrm{STF}(I_{ab})\,.$$

Here, M_{ab} is the Cartesian Symmetric and Trace Free (STF) mass quadrupole tensor, in the Newtonian limit equivalent to the usual potential coefficients (C_{lm}, S_{lm}) for $l = 2$.

For practical applications in the solar system usually one resorts to the first post-Newtonian approximation to Einstein's theory of gravity. There at least the construction of a post-Newtonian spin vector for one isolated body presents no problem (e.g., Fock 1959). In a series of papers Damour, Soffel and Xu (1991, 1992, 1993) formulated a new and improved approach to celestial mechanical problems within the post-Newtonian framework. Especially they gave a new definition of the spin (total intrinsic angular momentum) of a certain body A that is member of some gravitational N-body problem in the local frame comoving with body A. Also the PN dynamical equations for the evolution of the local spin vector was derived including the expression for the post-Newtonian torque. A Newtonian nutation theory with post-Newtonian torques was considered by Bizouard et al. (1992). For such a 'Newtonian plus a post-Newtonian torque' approach they found that the largest 'post-Newtonian corrections' to nutations in longitude and obliquity are given by

$$\Delta\Psi = 3 \times 10^{-7\prime\prime} \sin\Omega \, ; \qquad \Delta\epsilon = 4 \times 10^{-7\prime\prime} \cos\Omega$$

where Ω is the ascending node of the lunar orbit with respect to the ecliptic.

Post-Newtonian Tisserand axes and a post-Newtonian moment of inertia tensor were introduced by Klioner (1996). Finally, Klioner et al. (2001) formulated a useful generalization of the Newtonian Euler theory of a rigid Earth: the rigid multipole formalism. In that framework no assumptions on the local flow of matter are made; instead one assumes the post-Newtonian mass- and spin-multipole moments as well as the components of the moment of inertia tensor to rotate with some common angular velocity vector. If this formalism is actually consistent with Einstein's theory of gravity, however, is not clear. Likely no physical flow of matter obeying causality and the principles from relativity leads to rigidly rotating multipoles. This, however, would not reduce the usefulness of the formalism since it should serve only as intermediate step to model the real deformable Earth consistent with Einstein's theory of gravity.

In a series of papers Xu and coworkers laid the foundation for a relativistic description of elastic deformable rotating astronomical bodies (Xu et al., 2001, 2003, 2004). This work is based upon the introduction of a displacement field in the first post-Newtonian approximation to Einstein's theory of gravity. One starts with an isolated equilibrium configuration for the Earth, described in the GCRS and then considers small perturbations described by the displacement field resulting form the tidal forces and the elastic behaviour of the Earth. The formalism that was developed by Carter and Quintana (1972; Carter 1973) here serves as mathematical basis. In the first of these papers the basic post-Newtonian formalism is presented in Cartesian coordinates (Xu et al., 2001), the second rewrites the fundamental equations in spherical coordinates and discusses junction conditions for the Earth's surface and for internal layers. In the third paper first for the non-rotating ground state the relevant equations are expanded in terms of generalized spherical harmonics (scalar-, vector- and tensor spherical harmonics) so that the original system of partial differential equations reduces to an infinite system of ordinary differential equations for the expansion coefficients. Usually such a system serves as basis for numerical integrations as they were performed in the Newtonian case by Wahr (1982), Schastok (1997) and Dehant and Defraigne (1997). It should be stressed that this line of research is far from being complete and has to be pursued in several directions. An expansion of relevant equations into generalized spherical harmonics is still lacking for a rotating ground state, which is necessary for the description of precession and nutation. Because of the complexity of equations it might be useful to study the orders of magnitude of the various post-Newtonian terms first and to continue the work with the largest relativistic terms only. It might even be possible to understand the origin and meaning of several of the larger post-Newtonian terms.

It must be confessed that this local approach still poses fundamental problems since the relation of the formalism with observed quantities related with Earth's rotation is completely unclear at the moment. Also unclear is the use of the formalism in the frame of a perturbative approach. In the Newtonian framework one usually starts with a theory for a rigid Earth and adds perturbations due to the Earth's elastic behaviour, due to angular momentum exchange between solid Earth and other subsystems such as the atmosphere, the oceans, the hydrosphere etc. by means of a transfer function. A corresponding post-Newtonian generalization without the employment of a transfer function, however, does not exist.

The formulation of a high precision Newtonian nutation theory for a rigid Earth's model has been pursued by several groups: Souchay and Kinoshita (1996; REN2000), Bretagnon et al. (1997, 1998 (SMART97)) and Roosbeek and Dehant (1998; RDAN97). Actually, such Newtonian rigid Earth nutation series serve as basis for more realistic nutation series including the IAU2000 nutation series.

As mentioned above a useful generalization of this approach to the post-Newtonian framework would be the model of rigidly rotating multipoles. Such a model has already been worked out in detail (Klioner et al., 2001) and the derivation of a corresponding nutations series is feasible and highly desirable. The GCRS with its timescale $T =$ TCS will be the fundamental reference system.

In a first step the problem of constants and initial conditions for the relativistic equations of rotational motion should be investigated. The corresponding Newtonian initial conditions are discussed by Bretagnon et al., (1997, 1998). The following constants are relevant here: H_d: dynamical ellipticity of the Earth; a_E: its equatorial radius; masses of the Earth (M_E), Sun, Moon and planets; (A, B, C): principle moments of inertia of the Earth; (C_{lm}, S_{lm}): potential coefficients (mass multipole moments) of the Earth and the angle α describing the location of the principle axes with respect to the terrestrial $\hat{X}$-axis.

2. RELATIVITY IN THE PROBLEM OF ASTRONOMICAL REFERENCE SYSTEMS

For practical applications in the solar system the Barycentric Celestial Reference System BCRS with coordinates (t, x^i) and a corresponding Geocentric Celestial Reference System GCRS with coordinates (T, X^α) has been introduced. The properties of these two fundamental reference systems have been discussed exhaustively (e.g., Soffel et al., 2003). Still under discussion is the problem of orientation of spatial axes of the BCRS. Usually it is tacitly assumed that this orientation is given by the ICRF. Apart from these discussions it should be noted that the definitions of the BCRS and GCRS (or at least our understanding of these definitions) could be improved in several directions. One direction will be discussed below.

Present definitions of the BCRS assume the solar system to be isolated, i.e, one neglects cosmic matter outside of the solar system. Gravitational effects from neighbouring stars or galaxies can be considered as tidal terms in the same way that the Sun, Moon and planets are taken into account in the GCRS. These tidal terms have been estimated to be negligible for any applications in the solar system with current and foreseeable levels of accuracy, and thus neglected in the BCRS. Going to greater and greater distances at a certain distance scale the global geometry of the universe has to be considered. To include effects from the cosmological background is a more trickier task, and a new approach will be necessary here.

The apparently simple question whether the cosmological expansion happens also locally (that is, if a hydrogen atom or the Solar system also expand) is a very complicated one and presents still an unsolved problem. Starting from Einstein himself different authors got different answers using different arguments (see Bonnor (2000) for a review of recent progress). Certainly, the answer to this question crucially depends on our model of the matter in the universe and especially on the distribution of dark matter and "dark energy".

Recently (Soffel, Klioner, 2003; Klioner, Soffel, 2005), we have introduced the first "toy" version of the BCRS with cosmological terms in g_{00} and g_{ij}:

$$\begin{aligned} g_{00} &\approx -1 + \frac{2}{c^2}\, w(t, \mathbf{x}) - \frac{2}{c^4}\, w^2(t, \mathbf{x}) + \frac{1}{c^2}\, A_1\, |\mathbf{x}|^2, \\ g_{0i} &\approx -\frac{4}{c^3}\, w^i(t, \mathbf{x}), \\ g_{ij} &\approx \delta_{ij} \left(1 + \frac{2}{c^2}\, w(t, \mathbf{x}) + \frac{1}{c^2}\, B_1\, |\mathbf{x}|^2 \right), \end{aligned} \tag{1}$$

where $A_1 = -q\, H^2$ and $B_1 = -\frac{1}{2}\left(H^2 + \frac{k\, c^2}{a^2} \right)$, H is the Hubble constant, q is the decelation parameter of the Universe, k is the curvature parameter and a is the current "radius of the universe".

Here we neglected: (1) higher post-Newtonian terms $\mathcal{O}(c^{-5})$ in g_{00} and in g_{0i}, and $\mathcal{O}(c^{-4})$ in g_{ij} due to post-post-Newtonian (and higher order) effects from the solar system matter (they were also neglected in the BCRS adopted by the IAU); (2) higher-order cosmological terms $\mathcal{O}(|\mathbf{x}|^4)$; (3) any terms induced by the interaction of the cosmological fluid (including the cosmological constant) with the solar system matter. To derive the latter kind of terms is still an unsolved problem which appears to be quite tricky. That is why, we call this form of the BCRS with cosmological terms a "toy" version.

Current best estimates of the cosmological parameters give $A_1 \approx 3.2 \times 10^{-36}$ s^{-2} and $B_1 = -2.6 \times 10^{-36}$ s^{-2}. The estimates described in (Klioner, Soffel, 2005) shows that the direct dynamical effects of the cosmological terms in the solar system can be neglected totally. However, for light rays coming from very large distances cosmological effects play obviously an important role.

The BCRS metric with cosmological terms suggested above implies that the cosmological expansion has a certain influence on the properties of space-time within the solar system. This is certainly true for the terms coming from the cosmological constant Λ or vacuum energy since this energy source is present everywhere. Note that recent cosmological observations suggest that about 73% of the energy in the universe comes from that source. The applicability of the suggested BCRS metric to the 4% coming from the luminous matter and the 23% of the dark matter has to be investigated further.

2.1 Azimuth and elevation

For some applications a consistent post-Newtonian definition of geometrical horizon angles, azimuth A and elevation angle a might be useful. Note that such a definition does not yet exist and it is not clear if for some application post-Newtonian accuracy will be necessary (for standard applications such as telescope pointing obviously not).

Post-Newtonian horizon angles can be defined by introducing three orthonormal (with respect to the GCRS metric) horizon vectors that we call: horizon-(normal), south- and east-vector. A definition involving a simple geometrical horizon-vector, pointing in radial coordinate direction, could read like this. Start from the GCRS and consider the topocenter at time $T_0 = TCG_0$. Transform the spatial GCRS coordinates X^a to new topocentric coordinates $\hat{X}^a$ only by change of orientation and origin such that the $\hat{Z}$-axis runs through the topocenter, the $\hat{X}$-axis points 'southwards' and the $\hat{Y}$-axis 'eastwards' in a coordinate sense. Next transform to local proper coordinates in the same way that we transformed from BCRS to GCRS coordinates, where the geocenter now is replaced by the topocenter. Such a transformation defines a set of instantaneous tetrad vectors attached to the topocenter and the three spatial vectors are the vectors denoted by 'horizon-(normal), south- and east-vector'. By a suitable projection of the tangent vector to some incident light-ray onto the horizon-vector and the south-vector the two horizon angles, azimuth and elevation can be defined rigorously.

3. REFERENCES

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