INFLUENCE OF THE MULTIPOLe MOMENTS OF A GIANT PLANET ON THE PROPAGATION OF LIGHT: APPLICATION TO GAIA

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ABSTRACT. Approved space astrometry missions, like GAIA and SIM, are aimed to measure positions and/or parallaxes of celestial objects with an accuracy of 1-10 microarcseconds (µas). At such a level of accuracy, it will be indispensable to take into account the influence of the multipole structure of the giant planets (mainly Jupiter and Saturn) on the gravitational light deflection. Using the Nordtvedt-Will parametrized post-Newtonian formalism, we present an algorithmic procedure enabling to determine this influence on a light ray connecting two points located at a finite distance.

1. INTRODUCTION

Two major space astrometry missions, GAIA and SIM, are planned to be launched in the next years. The accuracy in the measurements of positions and/or parallaxes of celestial objects is expected to attain a level of 1-10 µas. In this context, we have to describe precisely the propagation of light inside and outside Solar System in a fully relativistic framework. By the time of Hipparcos mission, it was sufficient to consider the light deflection due to a static spherically symmetric Sun. Now, at the level of the µas accuracy, it is necessary to take into account the masses of the planets, as well as the higher multipole moments of those among them which are the most massive ones (Jupiter, Saturn, Uranus and Neptune). Table 1 gives the order of magnitude of the different contributions to the bending of a light ray propagating in Solar System. It is seen that for Jupiter, e.g., the effects of the multipole moments $J_2$ and $J_4$ may amount to 240 µas and 10 µas for a grazing light ray, respectively. So these effects must be taken into account in GAIA mission.

To take into account these intricate effects, several studies have been performed in the last decade. The first general relativistic model of positional observations at the level of 1 µas in space was proposed by Klioner & Kopeikin (1992), where gravitating bodies are considered as mass monopoles moving with constant velocities. More recently, a complete analytical description of the light propagation in the field of arbitrarily moving spinning mass monopoles bodies has been found by Kopeikin & Schäfer (1999) and Kopeikin & Mashhoon (2002) in the first post-Minkowskian approximation. For treating the particular problem of the multipole structure of celestial bodies, Hellings (1986) recommended to use the post-Newtonian formulas for the light propagation in the field of a motionless body and to introduce the position of each gravitating

204
Table 1: Gravitational bending of light rays in Solar System. Here $\delta_{\text{pn}}$ and $\delta_{\text{ppn}}$ are the post-Newtonian and the post-post-Newtonian effects due to the spherically symmetric field of the body, $\delta J_2$, $\delta J_4$ and $\delta J_6$ are the effects due to multipole moments $J_2$, $J_4$ and $J_6$, respectively, and $\delta R$ is the gravitomagnetic deflection. Each effect is evaluated for a grazing light ray. Unit is $\mu\text{as}$.

<table>
<thead>
<tr>
<th>Body</th>
<th>$\delta_{\text{pn}}$</th>
<th>$\delta J_2$</th>
<th>$\delta J_4$</th>
<th>$\delta J_6$</th>
<th>$\delta R$</th>
<th>$\delta_{\text{ppn}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>$1.752 \times 10^6$</td>
<td>1</td>
<td>–</td>
<td>–</td>
<td>0.7</td>
<td>11</td>
</tr>
<tr>
<td>Mercury</td>
<td>83</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
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<tr>
<td>Venus</td>
<td>493</td>
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<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Earth</td>
<td>574</td>
<td>0.6</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Moon</td>
<td>26</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Mars</td>
<td>116</td>
<td>0.2</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Jupiter</td>
<td>16270</td>
<td>240</td>
<td>10</td>
<td>$\geq 0.1$</td>
<td>0.2</td>
<td>–</td>
</tr>
<tr>
<td>Saturn</td>
<td>5780</td>
<td>95</td>
<td>6</td>
<td>$\geq 0.1$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Uranus</td>
<td>2080</td>
<td>8</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Neptune</td>
<td>2533</td>
<td>10</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
used). The quantity $|a|$ stands for the ordinary Euclidean norm of $a$. In what follows, greek indices run from 0 to 3, and latin indices run from 1 to 3.

2. ASTROMETRIC ANGLE WITHIN THE POST-NEWTONIAN APPROXIMATION

Let $\Gamma_1$ and $\Gamma_2$ be two light rays emitted at point $x_{A_1} = (ct_{A_1}, \mathbf{x}_{A_1})$ and $x_{A_2} = (ct_{A_2}, \mathbf{x}_{A_2})$ respectively and simultaneously received by an observer $B$ located at point $x_B = (ct_B, \mathbf{x}_B)$. Let $u$ be the unit 4-velocity of this observer. Denote by $l^{(1)}$ and $l^{(2)}$ the vectors tangent at $x_B$ to $\Gamma_1$ and $\Gamma_2$, respectively. Since $l^{(1)}$ and $l^{(2)}$ are null vectors, the angle $\phi$ between these rays as measured by the observer $B$ is given by

$$\cos \phi = 1 - \frac{l^{(1)} . l^{(2)}}{(u . l^{(1)})(u . l^{(2)})}_B.$$  

This formula holds in any gravitational field. Using the quasi-Cartesian coordinates system $(x^\alpha)$ introduced in section 1, Equation (1) may be explicitly written as

$$\cos \phi = 1 - \frac{g^{00} + g^{0i} \left( \frac{l^{(1)}_i}{l^0} + \frac{l^{(2)}_i}{l^0} \right) + g^{ij} \frac{l^{(1)}_i l^{(2)}_j}{l^0}}{(u^0)^2 \left( 1 + \frac{v^i}{c l^0} \right) \left( 1 + \frac{v^j}{c l^0} \right)}_B,$$

where $v^i = dx^i/dt$ is the coordinate velocity of the observer. In Le Poncin-Lafitte & al. (2004), we showed that the ratio $l_i/l_0$ can be explicitly determined when the time transfer functions are known. Let us recall that in general the travel time $t_B - t_A$ of a photon between an emission point $(ct_A, \mathbf{x}_A)$ and a reception point $(ct_B, \mathbf{x}_B)$ may be considered as a function of $t_A, \mathbf{x}_A$ and $\mathbf{x}_B$, or as a function of $t_B, \mathbf{x}_A$ and $\mathbf{x}_B$, so that we can put

$$t_B - t_A = T_e(t_A, \mathbf{x}_A, \mathbf{x}_B) = T_r(t_B, \mathbf{x}_A, \mathbf{x}_B),$$

where $T_e$ and $T_r$ may be called the emission and reception time transfer functions, respectively. We proved in the above-mentioned paper that the ratio $l_i/l_0$ is given at reception point $x_B$ by the relation

$$\left( \frac{l_i}{l_0} \right)_B = -c \frac{\partial T_e}{\partial x^i_B} = -c \frac{\partial T_r}{\partial x^i_B} \left[ 1 - \frac{\partial T_e}{\partial t_B} \right]^{-1}.$$

In what follows, we use the post-Newtonian approximation, so that the metric tensor may be written as

$$g_{00} = 1 - \frac{2W}{c^2} + O \left( \frac{1}{c^4} \right),$$

$$\{g_{0i}\} = \{h_{0i}\} = \vec{h} = O \left( \frac{1}{c^4} \right),$$

$$g_{ij} = \left( 1 + 2\gamma \frac{W}{c^2} \right) \eta_{ij} + O \left( \frac{1}{c^4} \right),$$

where $W = U + O(1/c^2)$, $U$ being the Newtonian-like potential of the body. For a light ray emitted at point $x_A$ and received at point $x_B$, we may write

$$\frac{l_i}{l_0} = -N^i + \Delta_i,$$
3. TIME TRANSFER AND LIGHT DEFLECTION

Let us apply these results to a light ray propagating in the field of an isolated, axisymmetric body. We suppose that the gravitational effects of the internal angular momentum of the body may be neglected. So we consider that the gravitational field is static. The center of mass of the body being taken as the origin O of quasi-Cartesian coordinates \((x)\), we choose the axis of symmetry as the \(x^3\)-axis. We put \(r = |x|, r_A = |x_A| \) and \(r_B = |x_B|\). We denote by \(k\) the unit vector along the \(x^3\)-axis and we consider only the case where all points of the segment joining \(x_A\) and \(x_B\) are outside the body. We denote by \(r_e\) the radius of the smallest sphere centered on O and containing the body (for celestial bodies, \(r_e\) is the equatorial radius). We assume the convergence of the multipole expansions formally derived below at any point outside the body, such that \(r > r_e\). On the above-mentioned assumptions, the two time transfer functions \(T_e\) and \(T_c\) reduce to a single function \(T(x_A, x_B)\) which may be expanded as

\[
T(x_A, x_B) = \frac{1}{c} R_{AB} + (\gamma + 1) \frac{GM}{c^3} R_{AB} F(0, x_A, x_B) + \sum_{n=2}^{\infty} T_{W,J_n}(x_A, x_B) + O \left( \frac{1}{c^7} \right),
\]

where \(F(x, x_A, x_B)\) is defined by

\[
F(x, x_A, x_B) = \frac{1}{R_{AB}} \ln \left( \frac{|x - x_A| + |x - x_B| + R_{AB}}{|x - x_A| + |x - x_B| - R_{AB}} \right),
\]

and \(T_{W,J_n}\) is the contribution of the mass multipole moment \(J_n\), given by

\[
T_{W,J_n}(x_A, x_B) = - (\gamma + 1) \frac{GM}{c^3} J_n r_e^n R_{AB} \frac{\partial^n}{\partial(x^3)^n} F(x, x_A, x_B) |_{x=0}.
\]

Calculating explicitly the successive derivatives of \(F(x, x_A, x_B)\), we find

\[
T_{W,J_n}(x_A, x_B) = - (\gamma + 1) \frac{GM}{c^3} J_n r_e^n \sum_{m=1}^{n} (-1)^m (m - 1)! \]

\[
\times \frac{(r_A + r_B + R_{AB})^m - (r_A + r_B - R_{AB})^m}{(r_A r_B + |x_A - x_B|)^m} \]

\[
\times \sum_{i_1 i_2 \ldots i_{n-m+1}} \frac{1}{i_1! i_2! \cdots i_{n-m+1}!} \prod_{l=1}^{n-m+1} \left[ \frac{1}{r_A^{l-1}} C_l^{(-1/2)} \left( \frac{k.x_A}{r_A} \right) + \frac{1}{r_B^{l-1}} C_l^{(-1/2)} \left( \frac{k.x_B}{r_B} \right) \right]^{i_l},
\]

where

\[
N^i = \frac{x^i_B - x^i_A}{R_{AB}}, \quad R_{AB} = |x_B - x_A|
\]

and \(\Delta_i\) is the relativistic contribution to the light deflection. As a consequence, Eq. (2) becomes

\[
\cos \phi = 1 - \frac{1 - \frac{v^2}{c^2} (1 - \frac{\sqrt{c^2 - v^2}}{c} N(1)) (1 - \frac{\sqrt{c^2 - v^2}}{c} N(2)) \left( \frac{1}{c^7} \right)}{\left( \frac{1}{c^7} \right)}
\]

where

\[
N(1) = \frac{x^1_B - x^1_A}{|x_B - x_A|}, \quad N(2) = \frac{x^2_B - x^2_A}{|x_B - x_A|},
\]

Let us note that Eq. (9) holds even if the gravitational field is not stationary.
where $C_l^{(-1/2)}$ denote the Gegenbauer polynomial of degree $l$ with parameter $-1/2$ (see Abramowitz and Stegun 1970) and $\sum'$ is a summation over all positive integers $i_1, i_2, \ldots, i_{n-m+1}$, solutions to the linear system

$$i_1 + 2i_2 + \ldots + (n-m+1)i_{n-m+1} = n, \quad i_1 + i_2 + \ldots + i_{n-m+1} = m.$$ 

We are now in a position to determine the covariant components of the vector tangent to the light ray emitted at point $x_A$ and received at point $x_B$. Applying Eqs. (4), (11) and (14), and then noting that one may set $t_0 = 1$ along the ray since the gravitational field is static, we find at point $x_B$

$$(l)_B = -N + (l^W)_B,$$

where

$$(l^W)_B = -(\gamma + 1) \frac{GM}{c^2} \left( \frac{r_A + r_B}{r_A r_B} - \frac{r_A + r_B + R_{AB}}{r_A r_B - R_{AB}} \right)^{m+1} + \sum_{n=2}^{\infty} (l^{W,J_n})_B(x_A, x_B),$$

with

$$(l^{W,J_n})_B(x_A, x_B) = (\gamma + 1) \frac{GM}{c^2} J_n r_e^n \left\{ \sum_{m=1}^{n} (-1)^{m+1} m! \times \left[ \frac{n_B - N}{(r_A + r_B - R_{AB})^{m+1}} - \frac{n_B + N}{(r_A + r_B + R_{AB})^{m+1}} \right] \sum' \frac{1}{i_1! i_2! \ldots i_{n-m+1}!} \prod_{l=1}^{n-m+1} D_{(l)}^{i_l} \right. \\
+ \sum_{m=1}^{n} (-1)^m (m-1)! \left[ \frac{1}{(r_A + r_B - R_{AB})^m} - \frac{1}{(r_A + r_B + R_{AB})^m} \right] \times \sum' \left[ \frac{1}{i_1! i_2! \ldots i_{n-m+1}!} \prod_{l=1}^{n-m+1} D_{(l)}^{i_l} \prod_{p=1, p \neq l}^{n-m+1} D_{(p)}^{i_p} \right] \times \left. \left[ n_B P_l \left( \frac{k \cdot x_B}{r_B} \right) - k P_{l-1} \left( \frac{k \cdot x_B}{r_B} \right) \right] \right\}. \tag{17}$$

$P_l$ being the Legendre polynomial of degree $l$, and $D_{(l)}$ being defined by

$$D_{(l)} = \frac{1}{r_A^{l-1}} c_l^{(-1/2)} \left( \frac{k \cdot x_A}{r_A} \right) + \frac{1}{r_B^{l-1}} c_l^{(-1/2)} \left( \frac{k \cdot x_B}{r_B} \right).$$

The contributions to angle $\phi$ due to $l^{W,J_n}$ are on current study now.

4. REFERENCES


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208
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