FORMULATION OF THE RELATION BETWEEN EARTH'S ROTATIONAL VARIATION EXCITATION AND TIME-VARIABLE GRAVITY

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1. INTRODUCTION

Slight temporal variations in both Earth's rotation and gravity field constantly take place as a result of mass transport in the Earth system, as governed by the conservation of angular momentum and Newton's gravitational law, respectively [e.g., *Chao et al.*, 2000]. These signals are observed on a routine basis by means of space geodetic techniques [e.g., *AGU*, 1993]. As geophysical observables, they can reveal insights about global climatic and geophysical processes and changes.

The gravity field is customarily and conveniently decomposed into its spherical harmonic components, or density multipoles of the gravitating body. For over two decades the satellite laser ranging technique has yielded the low-degree time-variable gravity (TVG) signals, as is the space mission GRACE in the last few years [*Tapley et al.*, 2004]. Among them, the lowest degree, i.e. degree-2, harmonic components of TVG are intimately related to the excitations of Earth's rotational variations (ERV), in the following sense:

The mass transports (besides external torques) that cause the Earth rotation to vary are referred to as geophysical excitations of the Earth rotation variations (ERV). As derived in *Munk and MacDonald* [1960], the excitation functions of ERV are the sum of two terms—the mass term and the motion term of angular momentum variation. While the motion term has no direct connection with gravity, the mass term does. As a vector, the ERV can be conveniently separated into two components: the (1-D) length-of-day variation (ΔLOD) and the (2-D) polar motion (PM). The mass term of the excitation of ΔLOD, under the conservation of the trace of Earth's inertia tensor, is directly proportional to the (degree, order) = (2,0) component of TVG, whereas the mass term of the PM excitation is directly proportional to the two (2,1) components of TVG.

The motion and mass terms (of angular momentum) are functionals of the mass transport. Combining the two independent measurements of ERV and TVG can thus reveal information about the separation of mass and motion terms in their contribution to the ERV excitation. Such information can be further compared to yet other independent sources of angular momentum estimation for the geophysical fluids, for example atmospheric and oceanic angular momenta, to provide constraints on the modeling of the latter.

Equally important, and perhaps geophysically more interesting, is the following: The gravity in general, and hence the TVG signal, comes from the whole Earth composed of both the mantle (including the crust) and the core (including outer and inner cores), whereas the excitation functions of ERV involve "mantle only" where the core are decoupled or only partially coupled to the mantle depending on the timescale and the type of coupling in question. This subtle difference, if detectable by exploiting the two independent measurements of ERV and TVG, can lead to insight into the Earth's dynamical processes as influenced by the strength and spectral dependence of the core-mantle coupling.

In this paper we shall develop the theoretical formulations that are relevant to conduct the study, and discuss their geophysical significance.

2. FORMULATION

In this section we build the complete formulation, by assembling elements already existing in the literature. The development consists of: (i) the relation between the Earth's gravitational harmonic components (in terms of the Stokes coefficients) and the mass distribution (in terms of multipole moments) [*Chao and Gross*, 1987, see also *Chao*, 1994; 2005]; (ii) as a special case of (i) the relation of the degree-2 Stokes coefficients with the quadrupole moments or the inertia tensor of the Earth, known as the generalized MacCullaugh formula [*Chao and Gross*, 1987]; (iii) the corresponding relation with respect to the temporal variations of (ii) [*Chao and Gross*, 1987; *Chao et al.*, 1987; *Chao*, 1994]; and (iv) the relation, following (iii), between the degree-2 Stokes coefficients of TVG and the excitation function of ERV [*Chao and Gross*, 1987]. It should be noted that the formulas for (i) – (iii) are exact, while those for (iv) are approximations. For the latter, the basic linearized theory for the excitation of ERV was developed by *Munk and MacDonald* [1960], and later specialized to exclude the core's participation by *Barnes et al.* [1983]. The relevant references from the literature are given as above, and will not be repeated below.

2.1 Degree-2 Stokes Coefficients and Inertia Tensor

Satisfying the Laplace equation, the external gravity potential field U produced by an arbitrary gravitating body (say the Earth) has a closed-form solution customarily expressed as a sum of spherical harmonic components in the spherical coordinates $\mathbf{r}$ = (radius r, co-latitude θ, longitude λ):

$$U(\mathbf{r}) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} P_{nm}(\cos\theta)\ \left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right) \tag{1}$$

[e.g., *Kaula*, 1966], where G is the gravitational constant, P_{nm} is the 4π-normalized Legedre function of degree n (= 0,1,2,...,∞) and order m(= 0,1,2,...,n), M is the mass of the gravitating body (which will be specified as the whole Earth or the mantle only, as the case may be, in the below). Referring to a, a length parameter conveniently chosen to be the mean equatorial radius of the Earth, the dimensionless coefficients C_{nm} and S_{nm} are known as the (normalized) Stokes coefficient of degree n and order m. The set of Stokes coefficients constitutes quantitatively what is a gravity field model.

Comparing Equation (1) with the multipole expansion of Uaccording to Newton's gravitational law [e.g., *Jackson*, 1975], one sees that the Stokes coefficients are simply normalized multipoles of the body with internal density distribution $\rho(\mathbf{r})$:

$$C_{nm} + iS_{nm} = \frac{1}{(2n+1)\,Ma^n} \iiint \rho(\mathbf{r})\ r^n\ P_{nm}(\cos\theta\ )\ \exp(im\lambda)\ \ dV \tag{2}$$

In particular, for degree n=2, as the right-side quadrupole moments are closely related to the inertia tensor $\mathbf{I}$ of the body, Equation (2) amounts to the generalized MacCullaugh formula:

$$
\begin{aligned}
C_{20} &= (I_{xx} + I_{yy} - 2I_{zz})/(2\sqrt{5}Ma^2) \\
C_{21} &= -\sqrt{3}I_{zx}/(\sqrt{5}Ma^2) \\
S_{21} &= -\sqrt{3}I_{yz}/(\sqrt{5}Ma^2) \\
C_{22} &= \sqrt{3}(I_{yy} - I_{xx})/(2\sqrt{5}Ma^2) \\
S_{22} &= -\sqrt{3}I_{xy}/(\sqrt{5}Ma^2)
\end{aligned} \tag{3}
$$

where expressed in the terrestrial Cartesian coordinates (where z-axis points to the mean North pole, and the x- and y-axes lie on the equatorial plan pointing, respectively, to the Greenwich Meridian and the 90°E Longitude):

$$
I = \begin{bmatrix} \int (y^2+z^2)\,\rho\, dV = I_{xx} & -\int xy\,\rho\, dV = I_{xy} & -\int zx\,\rho\, dV = I_{zx} \\ I_{xy} & \int (z^2+x^2)\,\rho\, dV = I_{yy} & -\int yz\,\rho\, dV = I_{yz} \\ I_{zx} & I_{yz} & \int (x^2+y^2)\,\rho\, dV = I_{zz} \end{bmatrix} \tag{4}
$$

The integrals are over the volume of the whole Earth or "mantle only", as the case may be. Note that there are 6 elements in **I** whereas only 5 degree-2 Stokes coefficients or quadrupole moments. The knowledge about the latter is insufficient to determine completely the former. This is a manifestation of the well-known non-uniqueness of the gravitational inversion [e.g., *Chao*, 2005]. We mention that the dynamic oblateness of the Earth is defined (for historical reasons) as $J_2 = -\sqrt{5}\, C_{20}$.

When a mass redistribution, or a change in the density function $\rho(\mathbf{r})$ takes place, the Stokes coefficients in (2) will change accordingly, and so will the inertia tensor (4). The changes can be evaluated using either the Lagrangian or Eulerian approaches (depending on the convenience dictated by the form of the data); but however they are evaluated, we now have the following relation:

$$
\begin{aligned}
\Delta C_{20} &= (\Delta I_{xx} + \Delta I_{yy} - 2\Delta I_{zz})/(2\sqrt{5}Ma^2) \\
\Delta C_{21} &= -\sqrt{3}\Delta I_{zx}/(\sqrt{5}Ma^2) \\
\Delta S_{21} &= -\sqrt{3}\Delta I_{yz}/(\sqrt{5}Ma^2) \\
\Delta C_{22} &= \sqrt{3}(\Delta I_{yy} - \Delta I_{xx})/(2\sqrt{5}Ma^2) \\
\Delta S_{22} &= -\sqrt{3}\Delta I_{xy}/(\sqrt{5}Ma^2) \\
\Delta T &= \Delta I_{xx} + \Delta I_{yy} + \Delta I_{zz}
\end{aligned} \tag{5}
$$

where in this paper Δ means "time variation in", so that the quantity following Δ is a function of time. Note here we have appended a formula for the quantity $T=$ Tr(**I**), the trace of the inertia tensor, anticipating its usage later.

2.2 Length-of-Day Change (ΔLOD)

If the body is under rotation, then any change in $\rho(\mathbf{r})$ and hence in **I** will induce changes in the rotation, as governed by the conservation of angular momentum. The conservation of the z-component of the angular momentum vector dictates that the z-component of the rotation vector, or consequently the ΔLOD if the timescale under consideration is longer than a day, obeys the equation:

$$
\Psi_{z[mass]} = \frac{\Delta\Omega_{mass}}{\Omega} = -\frac{\Delta \mathrm{LOD}_{mass}}{\mathrm{LOD}} = -\frac{\Delta I_{zz}}{I_{zz}} = \frac{2\sqrt{5}Ma^2\Delta C_{20} - \Delta T}{3I_{zz}} \tag{6}
$$

where Ψ_z is the (dimensionless) excitation function of ΔLOD, and the last equality is readily derivable from Equation (5). The subscript "mass" denotes "the part due to the mass term".

Apart from external torques, two parts of excitation contribute to the observed ΔLOD: a "mass" term due to mass redistribution as described here, and a "motion" term arising from relative angular momentum exchange with other parts of the Earth (via internal torques). For example, in the atmosphere or the oceans under the Eulerian approach, the mass term (which can be readily derived from above) and the motion term are given approximately by [*Barnes et al.*, 1983]:

$$\Psi_{z[mass]} = -\frac{0.70\, a^4}{g\, I_{zz}} \iint p \sin^3\theta d\theta d\lambda \tag{7}$$

$$\Psi_{z[motion]} = -\frac{1.00\, a^3}{\Omega\, g\, I_{zz}} \iiint u \ \sin^2\theta \ d\theta d\lambda \ dp \tag{8}$$

where $g = GM/a^2$ is the mean gravitational acceleration of the Earth, p is the surface pressure field, and u is the east-velocity field of mass transport (for example, of atmospheric winds or oceanic currents). The numerical coefficient $0.70 = 1{+}k_2$' accounts for the elastic yielding of the Earth as a result of the mass loading on the solid Earth, where k_2' is Earth's load Love number of degree 2. The mass term is related to TVG; the motion term has no direct relationship with gravity.

2.3 Polar Motion (PM) Excitation

Similarly, the conservation of the (equatorial) x-y components of the angular momentum vector results in:

$$\Psi_{mass} = 1.43 \frac{\Delta I_{zx} + i\Delta I_{yz}}{I_{zz} - I_{xx}} = -1.43 \frac{\sqrt{5}\, Ma^2(\Delta C_{21} + i\Delta S_{21})}{\sqrt{3}\,(I_{zz} - I_{xx})} \tag{9}$$

Here the complex-valued $\Psi = \Psi_x{+}i\Psi_y$ is an abbreviation for the (non-dimensional) polar motion excitation function whose real part is the x-component and the imaginary part the y-component. The factor 1.43 accounts for the dynamic feedback of the elastic rotational deformation itself that lengthens the period of the Chandler wobble from the rigid-Earth value of about 10 months to the observed 14 months.

Similarly as above, in the Eulerian approach for the atmosphere and oceans, we have approximately [*Barnes et al.*, 1983]:

$$\Psi_{mass} = -\frac{1.00\, a^4}{(I_{zz} - I_{xx})\, g} \iint p \ \cos\theta \ \sin^2\theta \ e^{i\lambda} d\theta d\lambda \tag{10}$$

$$\Psi_{motion} = -\frac{1.43\ a^3}{(I_{zz} - I_{xx})\Omega\, g} \iiint (u\cos\theta \ + iv) \ e^{i\lambda} \ \sin\theta d\theta d\lambda\, dp \tag{11}$$

where v is the north-velocity field of the mass transport, and an axial symmetric approximation of $I_{xx} = I_{yy}$ has been made. Here the numerical coefficients 1.00 results from the product of 1.43 with $1{+}k_2$' arising from the mass loading effect.

The "observed" PM excitation Ψ is related to the observed PM P via:

$$\Psi = P - (1/i\omega_c) \quad \partial_t P \tag{12}$$

where ω_c is the resonance Chandler wobble frequency of the Earth.
Equation (12) constitutes a deconvolution relation, because, when solved, it states that P is the temporal convolution of Ψ with the free Chandler wobble.

2.4 Rotation-Derived and the Gravity-Derived Quantities

Equations (6) and (9) can be re-written respectively as

$$\text{rotation−derived } \Delta C_{20} = \frac{3\, I_{zz}\, \Psi_{z[mass]} + \Delta T}{2\sqrt{5} M a^2} \tag{13}$$

$$\text{rotation−derived } (\Delta C_{21} + i\,\Delta S_{21}) = \frac{-\sqrt{3}\,(I_{zz} - I_{xx})}{1.43\sqrt{5}\, M a^2} \Psi_{mass} \tag{14}$$

The goal of this research is then to compare the rotation-derived quantities (13) and (14) with the independent, corresponding gravity-derived counterparts, and to extract geophysical information and insights in the process. A complete study would require extensive data analysis, which will await future effort. In the next section we will discuss the geophysical significances.

3. GEOPHYSICAL DISCUSSIONS

3.1 Mass term of ERV excitation function

For the last quarter century the space geodetic techniques of satellite-laser-ranging (SLR) and very-long-baseline interferometry (VLBI), and the more recent addition of the Global Positioning System (GPS), have been obtaining precise measurements of ERV, in both ΔLOD and PM. The total excitation function derived from the ERV observations contains both mass and motion terms: $\Psi_z = \Psi_{z[mass]} + \Psi_{z[motion]}$, and $\Psi = \Psi_{mass} + \Psi_{motion}$ (where Ψ is derived by means of Equation 12). They must be stripped of the motion-term contribution to become comparable with the corresponding gravity-derived quantities which are related to the mass term only.

This can in principle be accomplished by introducing, and subtracting off, independent estimates for the motion terms for the geophysical fluids, including atmosphere, oceans, land hydrology, core, etc. [e.g., *Chao et al.*, 2000]. It is well known that the (zonal) motion terms contribute dominantly in the case of ΔLOD or Ψ_z excitation. On interannual to weekly timescales including the seasonal periodicities, the motion term of the atmospheric angular momentum (AAM) accounts for the majority of ΔLOD [e.g., *Salstein et al.*, 1993], while some secondary contributions come from the motion term of the non-tidal oceanic angular momentum (OAM) [e.g., *Marcus et al*, 1998; *Johnson et al.*, 1999; *Gross*, 2003]. The large decadal fluctuation in ΔLOD arises from the motion term in the core angular momentum (CAM) [e.g., *Holme and Whaler*, 2001]. The strong motion terms of the ocean tidal angular momentum are of much shorter periods than of interest here. The hydrological angular momentum and the solid-Earth bodily tides have negligible motion terms. The similar is true with respect to the PM excitation, although the contribution of the motion terms is no longer dominant, but rather comparable or smaller relative to the mass terms.

It should be noted that subtracting the motion-term contributions of the geophysical fluids from the observed total ERV excitation function has the undesirable consequence of magnifying the noise to signal ratio in the residual mass term. Furthermore, any remnant motion-term contributions that are not removed completely become sources of error.

The end products of the removal of motion terms are thus the $\Psi_{z[mass]}$ and Ψ_{mass} needed in Equations (13) and (14).

3.2 Does ΔT vanish?

We note in Equation (13) the inclusion of the "extra" term ΔT (defined in Equation 5). Unless ΔT vanishes, its existence becomes troublesome when we try to relate ΔC_{20} with ΔLOD, because neither rotation nor gravity would observe it directly. Although an invariant under coordinate transformation, T can definitely vary with respect to time. An example is co-seismic dislocation [*Chao and Gross*, 1987].

However, for all practical purposes many Earth processes of mass transport do preserve T, rendering true $\Delta T \approx 0$. Possible examples include glacial isostatic adjustment [R. Peltier, personal communication, 2002] and mantle convection. More significantly, ΔT indeed vanishes on timescales that are dominated by an important class of mass redistribution processes – those taking place on the surface of the Earth, most notably in the form of air and water mass transports: It is easy to show that $\Delta T = 0$ under the conservation of the total surface mass, as long as the surface is (assumed) spherical [e.g., *Chao et al.*, 1987].

Thus, while there is no a priori reason why ΔT should vanish, one can enforce the simplification $\Delta T = 0$, so that Equation (11) becomes

$$\text{rotation}-\text{derived } \Delta C_{20} \quad \approx \quad \frac{3\ I_{zz}}{2\sqrt{5}Ma^2}\,\Psi_{z[mass]} \tag{15}$$

Any ΔLOD signal coming from sources for which $\Delta T \neq 0$ will thus contain such "contamination" when converted into the equivalent ΔC_{20}.

3.3 "Mantle Only" versus "Whole Earth"

Finally, we should consider a fundamental question pertaining to the exact meaning behind Equations (13-15) – whether they apply to the case of the "Whole Earth" or the "Mantle Only". These are dynamical scenarios that represent two extremes: The "Whole Earth" corresponds to 100% coupling of the core with the mantle in the ERV excitation process, and the "Mantle Only" corresponds to zero core-mantle coupling in the ERV excitation process. The reality presumably lies somewhere in between the extreme cases, but is a strong function of the timescale and the mechanisms at work, which may even distinguish between the axial and equatorial components. For example, it appears that the "Mantle Only" scenario is a reasonable approximation on timescales shorter than several years, longer than which it has been demonstrated that the CAM strongly affects ΔLOD [e.g., *Holme and Whaler*, 2001] – the transition from non-coupling to strong-coupling is thus around several years. Similar arguments apply to the PM excitation.

On the other hand, the gravity-derived quantities refers only to the "Whole Earth", as no mass can be "shielded" and not be observed gravitationally from outside. Therefore the corresponding rotation-derived and gravity-derived values of ΔC_{20} and $\Delta C_{21} + i\Delta S_{21}$ differ by the contribution of the core, the amount of which depends on the strength of the core-mantle coupling in the ERV excitation processes [e.g., *Dickman*, 2003].

Suppose we take, say, "Mantle Only" as the baseline case by letting all the quantities in Equations (13-15) assume mantle-only values (except that Ma^2 is the whole-Earth parameter merely serving as normalization factors). Then any observed departure of the gravity-derived quantities from the baseline rotation-derived quantities evaluated accordingly will in principle signify the departure of the reality from the underlying "Mantle Only" assumption for the ERV excitation processes. Such extraction of geophysical information awaits further investigations.

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4. REFERENCES

AGU, *Contributions of Space Geodesy to Geodynamics: Technology*, Geodynamics Series 25, ed. D. E. Smith and D. L. Turcotte, Amer. Geophys. Union, Washington DC, 1993.

Barnes, R.T.H., R. Hide, A.A. White, and C.A. Wilson, Atmospheric angular momentum fluctuations, length-of-day changes and polar motion. *Proc. Roy. Soc. Lond. A, 387*, 31-73, 1983.

Chao, B. F., and R. S. Gross, Changes in the Earth's rotation and lowdegree gravitational field induced by earthquakes, *Geophys. J. Roy. Astron. Soc., 91,* 569596, 1987.

Chao, B. F., W. P. O'Connor, A. T. C. Chang, D. K. Hall, and J. L. Foster, Snowload effect on the Earth's rotation and gravitational field, 1979-1985, *J. Geophys. Res., 92,* 94159422, 1987.

Chao, B. F., The Geoid and Earth Rotation, in *Geophysical Interpretations of Geoid*, ed. P. Vanicek and N. Christou, CRC Press, Boca Raton, 1994.

Chao, B. F., V. Dehant, R. S. Gross, R. D. Ray, D. A. Salstein, M. M. Watkins, and C. R. Wilson, Space geodesy monitors mass transports in global geophysical fluids, *EOS, Trans. Amer. Geophys. Union, 81,* 247-250, 2000.

Chao, B. F., On inversion for mass distribution from global (time-variable) gravity field, in press, *J. Geodynamics*, 2005.

Dickman S. R., Evaluation of "effective angular momentum function" formulations with respect to core-mantle coupling, *J. Geophys. Res., 108* (B3), 2150, doi:10.1029/2001JB001603, 2003.

Gross, R. S., I. Fukumori, and D. Menemenlis, Atmospheric and oceanic excitation of the Earth's wobbles during 1980-2000, *J. Geophys. Res.,* 108 (B8), 2370, 10.1029/2002JB002143, 2003.

Holme, R., and K. Whaler, Steady core flow in an azimuthally drifting reference frame, *Geophys. J. Int., 145*, 560-569, 2001.

Jackson, J. D., *Classical Electrodynamics*, 2nd ed., Wiley, New York, 1975.

Johnson, T. J., C. R. Wilson, and B. F. Chao, Oceanic angular momentum variability estimated from the Parallel Ocean Climate Model, 1988-1998, *J. Geophys. Res., 104*, 25183-25196, 1999.

Kaula, W. M., *Theory of satellite geodesy*, Blaisdell Publishing Co., Waltham, 1966.

Marcus, S. L., Y. Chao, J. O. Dickey, and P. Gegout, Detection and modeling of nontidal oceanic effects on Earth's rotation rate, *Science*, 281, 1656–1659, 1998.

Munk W. H., and G. J. F. MacDonald, *The Rotation of the Earth*, Cambridge Univ. Press, New York, 1960.

Salstein, D.A., D.M. Kann, A.J. Miller, and R.D. Rosen, The sub-bureau for Atmospheric Angular Momentum of the International Earth Rotation Service (IERS): A meteorological data center with geodetic applications, *Bull. Am. Met. Soc.*, 74, 67-80, 1993.

Tapley, B. D., S. Bettadpur, J. C. Ries, P. F. Thompson, and M. M. Watkins, GRACE measurements of mass variability in the Earth system, *Science, 305*, 503-505, 2004.