RELATIVISTIC INDIRECT THIRD–BODY PERTURBATIONS IN THE SMART EARTH'S ROTATION THEORY

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ABSTRACT. In neglecting by very small relativistic direct third–body perturbations it is possible to use formally Newtonian equations of the Earth's rotation in DGRSC, dynamically non–rotating geocentric ecliptical reference system with Terrestrial time TT as the time argument. To obtain relativistic indirect third–body perturbations in the SMART theory of the Earth's rotation one can use the relativistic formulae expressing the DGRSC position vectors referred to TT in terms of the VSOP BRSC (barycentric ecliptical reference system) position vectors referred to Barycentric dynamical time TDB. Then the right–hand members become functions of TT alone and one can use the standard SMART iteration techniques to obtain the relativistic contributions into three Euler angles relating ITRS and DGRSC.

1. INTRODUCTION

SMART97 (Bretagnon et al. 1997, 1998, 2003) represents the most accurate semi–analytical theory of rotation of the rigid Earth constructed so far. In the recent years Pierre Bretagnon, the principal author of this theory, was extending it for the case of the non–rigid Earth (Bretagnon 2002) using the transfer function of Mathews et al. (2002). SMART97 is a purely Newtonian theory. Bretagnon was going also to convert it into relativistic theory (Bretagnon and Brumberg 2003) but his death broke off this work. The aim of the present paper is to continue it.

One may find in literature relativistic equations of the Earth's equations of different type in dependence on adopted Earth's model (see Brumberg 1998; Klioner and Soffel 1998, 1999 and references therein). Instead of dealing with such complicated equations we prefer to start with by taking into account in SMART97 the relativistic indirect third–body perturbations as proposed by Bretagnon and Brumberg (2003). In doing so, we neglect by very small direct relativistic third–body perturbations. It enables us to retain the formally Newtonian differential equations of the Earth's rotation and to get the relativistic extension of SMART97 solution by applying in the right–hand members of these equations the four–dimensional transformation between geocentric and barycentric quantities. It leads to the main relativistic terms in the

Earth's rotation problem called by us the relativistic indirect third–body perturbations.

All expressions below are given in the post–Newtonian approximation within c^{-2} accuracy.

2. RS HIERARCHY

2.1 Barycentric and geocentric reference systems

We use below the hierarchy of relativistic reference systems (RSs) constructed in (Brumberg et al. 1996; Brumberg 1997; see also the detailed exposition by Bretagnon and Brumberg 2003) and illustrated by Fig. 1.

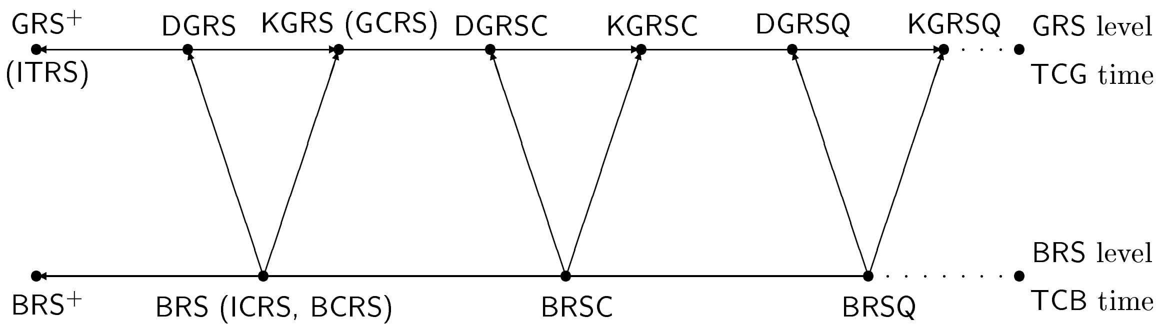

Figure 1: Barycentric and Geocentric Reference Systems (RSs)
B — barycentric, G — geocentric, C — ecliptical, Q — equatorial,
D — dynamical, K — kinematical, + — rotating (BBG, 1996);
ICRS — International Celestial RS, ITRS — International Terrestrial RS (IERS);
BCRS — Barycentric Celestial RS, GCRS — Geocentric Celestial RS (IAU 2000)

The four reference systems of the barycentric (B) level referred to the barycentric coordinate time t =TCB (or to TDB in practice) are related by means of

$$[\mathrm{BRS}^+] = P(t)[\mathrm{BRSC}] = P(t)P_C[\mathrm{BRS}] = P(t)P_C P_Q^T[\mathrm{BRSQ}]. \qquad (1)$$

[ ] means here and below a triplet of the corresponding spatial coordinates.

The corresponding reference systems at the geocentric (G) level referred to the geocentric coordinate time u =TCG (or to TT in practice) are related by means of

$$[\mathrm{GRS}^+] = \underset{1}{\hat{P}}(u)[\mathrm{DGRSC}] = \underset{1}{\hat{P}}(u)P_C[\mathrm{DGRS}] = \underset{1}{\hat{P}}(u)P_C P_Q^T[\mathrm{DGRSQ}], \qquad (2)$$

$$[\mathrm{GRS}^+] = \underset{0}{\hat{P}}(u)[\mathrm{KGRSC}] = \underset{0}{\hat{P}}(u)P_C[\mathrm{KGRS}] = \underset{0}{\hat{P}}(u)P_C P_Q^T[\mathrm{KGRSQ}]. \qquad (3)$$

Kinematically (K) or dynamically (D) non–rotating GRSs are distinguished by subscript q, i.e. $q = 0$ for K versions and $q = 1$ for D versions. K and D versions of GRSs are related by means of

$$[\mathrm{KGRS}] = (E - c^{-2}F)[\mathrm{DGRS}], \qquad [\mathrm{KGRSC}] = (E - c^{-2}F_C)[\mathrm{DGRSC}],$$

$$[\mathrm{KGRSQ}] = (E - c^{-2}F_Q)[\mathrm{DGRSQ}] \qquad (4)$$

and

$$\underset{1}{\hat{P}}(u) = \underset{0}{\hat{P}}(u)(E - c^{-2}F_C). \qquad (5)$$

Here F stands for the geodetic rotation matrix (its expression vanishing for J2000.0 is given by Bretagnon and Brumberg 2003) whereas F_C and F_Q represents its analogues for the ecliptical (C) and equatorial (Q) GRSs, respectively, i.e.

$$F_C = P_C F P_C^T, \qquad F_Q = P_Q F P_Q^T. \tag{6}$$

The constant rotation matrices P_C, P_Q used in this paper are taken in accordance with DE403, i.e.

$$P_C = D_1(\varepsilon) D_3(\chi), \qquad P_Q = D_3(\chi) \tag{7}$$

where $\varepsilon = 23°26'21.40928''$, $\chi = -0.05294''$. The slightly different values are proposed in (Bretagnon 2002; Bretagnon et al. 2003) for future analytical planetary ephemerides. The problem of consistency of the existing analytical and numerical planetary and lunar theories as well as of the Earth's rotation theory with the hierarchy of relativistic reference systems is not still completely solved.

Operations with rotation matrices are often replaced by operation with rotation vectors by using the relationships of the type

$$F^i = \tfrac{1}{2}\varepsilon_{ijk} F^{jk}, \qquad F^{ij} = \varepsilon_{ijk} F^k, \qquad \varepsilon_{ijk} = \tfrac{1}{2}(i-j)(j-k)(k-i). \tag{8}$$

$\underset{1}{\hat{P}}(u)$ represents the Earth's rotation matrix relating DGRSC and terrestrial matrix ITRS (designated here also as GRS$^+$). Since SMART97 is supposed to be constructed in DGRSC three Euler angles, $\underset{1}{\psi}$, $\underset{1}{\theta}$, $\underset{1}{\varphi}$, of matrix $\underset{1}{\hat{P}}(u)$ may be regarded as dynamical Earth orientation parameters (EOP). The analogous Euler angles $\underset{0}{\psi}$, $\underset{0}{\theta}$, $\underset{0}{\varphi}$, of matrix $\underset{0}{\hat{P}}(u)$ relating KGRSC and ITRS may be regarded as kinematical EOP. One has

$$\underset{q}{\hat{P}}(u) = D_3(\underset{q}{\varphi}) D_1(-\underset{q}{\theta}) D_3(-\underset{q}{\psi}) \qquad (q = 0,\, 1), \tag{9}$$

D_i being the elementary rotation matrices. The dynamical and kinematical Euler angles are related by the formulae

$$\underset{1}{\varphi} - \underset{0}{\varphi} = -\frac{c^{-2}}{\sin\theta}\left(F_C^1 \sin\psi + F_C^2 \cos\psi\right), \qquad \underset{1}{\theta} - \underset{0}{\theta} = c^{-2}\left(F_C^1 \cos\psi - F_C^2 \sin\psi\right),$$

$$\underset{1}{\psi} - \underset{0}{\psi} = c^{-2}\left[F_C^3 - \frac{\cos\theta}{\sin\theta}\left(F_C^1 \sin\psi + F_C^2 \cos\psi\right)\right] \tag{10}$$

(in the post–Newtonian approximation there is no need to distinguish between Newtonian and relativistic values in the relativistic right–hand members). These relationships have been actually used in SMART. In taking into account only the geodesic precession and nutation in narrow sense, one has $F_C^1 = F_C^2 = 0$ and, hence, $\underset{1}{\varphi} = \underset{0}{\varphi}$, $\underset{1}{\theta} = \underset{0}{\theta}$.

Note that to get the designations of the original papers on SMART (Bretagnon et al. 1997, 1998) one should put $\psi = -\psi$ and $\theta = -\omega$.

To complete the discussion of the RS hierarchy of Fig. 1 let us note that the Earth's rotation matrix relating GCRS and ITRS is determined in our notation as $T = \underset{0}{\hat{P}}(u) P_C$. The Earth's rotation in BRS may be described by the rotation matrix $P(t^*) = \underset{0}{\hat{P}}(u)$ where t^* is the solution of the relativistic time equation

$$u = t^* - c^{-2} A(t^*) \tag{11}$$

with the time function determined by

$$\dot{A}(t) = \tfrac{1}{2}\mathbf{v}_E^2 + \bar{U}_E(t, \mathbf{x}_E), \qquad \bar{U}_E(t, \mathbf{x}_E) = \sum_{A \neq E} \frac{GM_A}{r_{EA}} \tag{12}$$

with initial condition $A(t_0) = 0$, t_0 being the 1977 Origin (see the detailed discussion by Bretagnon and Brumberg 2003). However, rotating system BRS$^+$ is not used in practice.

2.2 Relativistic extension of SMART97 by using RS hierarchy

It is assumed that VSOP theories are constructed in BRSC with TDB as a time argument while SMART97 is considered in DGRSC with TT as a time argument. Therefore, in treating SMART in the relativistic framework the values of masses should be adequately adjusted taking into account that $(GM)_{\rm TDB}$ coefficients in VSOP and $(GM)_{\rm TT}$ coefficients in SMART are related by

$$(GM)_{\rm TT} = (1 + L_C)(GM)_{\rm TDB} \tag{13}$$

with the value of $L_C = 1.480826855667 \times 10^{-8}$ obtained with the VSOP solution (Bretagnon and Brumberg 2003). But this mass–adjustment is not made in the present work aimed to evaluate the influence of the relativistic indirect third–body perturbations. Indeed, the main perturbation factors in the right–hand members of the DGRSC equations of the Earth's rotation, are due to the action of the Sun (S) and the Moon (L). Initially, these right–hand members contain geocentric position vectors $\mathbf{w}_A$ for $A = S,\, L$. As proposed in (Bretagnon and Brumberg 2003) these geocentric vectors are to be expressed by virtue of BRSC↔DGRSC transformation in terms of BRSC quantities as follows:

$$w^i_A(u) = z^i_A(u) - z^i_E(u) + c^{-2}\left[\Lambda^i(t^*,\, \vec{r}_{AE}) + \mathbf{v}_E\mathbf{r}_{AE}v^i_{AE}\right], \tag{14}$$

with $\mathbf{x}_E$, $\mathbf{v}_E$, $\mathbf{x}_A$, $\mathbf{v}_A$ denoting BRSC coordinates and velocities of the Earth and the disturbing body, respectively, $\mathbf{r}_{AE} = \mathbf{x}_A - \mathbf{x}_E$, $\mathbf{v}_{AE} = \vec{v}_A - \mathbf{v}_E$ and

$$\Lambda^i(t,\, \mathbf{r}_{AE}) = \tfrac{1}{2}\mathbf{v}_E\mathbf{r}_{AE}v^i_E - q\varepsilon_{ijk}F^j r^k_{AE} + \bar{U}_E(t, \mathbf{x}_E)r^i_{AE} + \mathbf{a}_E\mathbf{r}_{AE}r^i_{AE} - \tfrac{1}{2}\mathbf{r}^2_{AE}a^i_E\,, \tag{15}$$

$\mathbf{a}_E$ being BRSC acceleration of the Earth. The moment t^* means here

$$\mathrm{TDB}^* = \mathrm{TT} + c^{-2}A_p \tag{16}$$

if time function $A(t)$ is represented in TDB as

$$A(t) = c^2 L_C t + A_p(t). \tag{17}$$

The function z^i_E representing the BRSC position of the Earth in terms of TCG or TT is given in our case by

$$(1 - L_C)z^i_E(\mathrm{TT}) = x^i_E(\mathrm{TDB}^*) = x^i_E(\mathrm{TT}) + c^{-2}A_p v^i_E + \ldots\,. \tag{18}$$

The function z^i_A is determined by the same formula by replacing E for A. The power–trigonometric time series for all functions occurring here are tabulated in (Bretagnon and Brumberg 2003).

Functions $x^i_E(\mathrm{TT})$, $x^i_A(\mathrm{TT})$ represent just VSOP series of the argument TDB taken for the moment TT. Therefore, they are expressed in terms of 11 fundamental arguments (mean longitudes of eight major planets and Delaunay arguments D, F, l of the lunar theory) representing now linear functions of TT. One more fundamental argument ϕ (the linear part of the expression for the Euler angle φ) is specific for SMART solution. In such a way, the right–hand members of the DGRSC equation of the Earth's rotation become functions of TT and may be solved by iterations just in Newtonian case (Bretagnon et al. 1997, 1998). In result we get the solution taking into account relativistic indirect third–body perturbations. Based on the dynamical solution for $\underset{1}{\psi}$, $\underset{1}{\theta}$, $\underset{1}{\varphi}$ one finds by means of (10) the kinematical solution $\underset{0}{\psi}$, $\underset{0}{\theta}$, $\underset{0}{\varphi}$ and then the astrometric Earth's rotation matrix $T = \underset{0}{\hat{P}}(u)P_C$ including now the main relativistic corrections.

3. RIGHT–HAND MEMBERS

Computation of the right–hand members of the Earth's rotation equations in the SMART theory (Bretagnon et al. 1997, 1998) is based on the VSOP series for $\mathbf{x}_{(C)A}(\mathrm{TDB})$ where (C) indicates the ecliptical system BRSC and A stands for the body A ($A = E$ for the Earth, $A = S$ for the Sun, $A = L$ for the Moon, etc.). In the original SMART theory referred to TDB the geocentric coordinates of the Sun and the Moon in DGRSC are treated just as the differences of the corresponding BRSC coordinates $\mathbf{x}_{(C)S} - \mathbf{x}_{(C)E}$ and $\mathbf{x}_{(C)L} - \mathbf{x}_{(C)E}$ referred to TDB. In the present work the equations of the Earth's rotation are referred to TT with using $\mathbf{w}_{(C)A}(\mathrm{TT})$ for the geocentric coordinates of the Sun ($A = S$) and the Moon ($A = L$). Considering the smallness of the planetary perturbations in the Earth's rotation problem all disturbing planets may be treated just as in the Newtonian case, i.e. by putting $\mathbf{w}_{(C)A}(\mathrm{TT}) \approx \mathbf{x}_{(C)A}(\mathrm{TDB}) - \mathbf{x}_{(C)A}(\mathrm{TDB})$ and TT≈TDB.

Starting from the VSOP values $\mathbf{x}_{(C)A}(\mathrm{TDB})$ for $A = E$, S, L and the ICRS coordinates of the Earth $\mathbf{z}_E(\mathrm{TT})$ resulted from the data of (Bretagnon and Brumberg 2003) we compute successively the series for the solar BRSC coordinates $\mathbf{z}_{(C)S}(\mathrm{TT})$, for the lunar BRSC coordinates $\mathbf{x}_{(C)L}(\mathrm{TDB}) - \mathbf{x}_{(C)E}(\mathrm{TDB})$ and $\mathbf{z}_{(C)L}(\mathrm{TT}) - \mathbf{z}_{(C)E}(\mathrm{TT})$, and finally for the solar DGRSC geocentric coordinates $\mathbf{w}_{(C)S}(\mathrm{TT})$ and for the lunar DGRSC geocentric coordinates $\mathbf{w}_{(C)L}(\mathrm{TT})$. All series are presented in the compact form adopted presently in VSOP, i.e.

$$x^i_A(t) = \sum_\alpha t^\alpha \left[\sum_k X^\alpha_{ik} \cos(\psi^\alpha_k + \nu^\alpha_k\, t) \right] \tag{19}$$

The time argument t is in fact either TDB or TT. The fundamental trigonometric arguments of the semi–analytical SMART series are given in Appendix A.

4. FINAL EXPANSIONS

Having got the relativistic coordinates $\mathbf{w}_{(C)A}(\mathrm{TT})$ ($A = S$, L) we perform iterations as described in (Bretagnon et al. 1997, 1998) to obtain the solution with taking into account the main relativistic indirect third–body perturbations. This solution is compared with the Newtonian SMART solution based on the Newtonian luni–solar coordinates $\mathbf{x}_{(C)A}(\mathrm{TDB}) - \mathbf{x}_{(C)E}(\mathrm{TDB})$ ($A = S$, L). Since in the Newtonian theory there is no difference between TT and TDB we may just compare the coefficients of the series (19) for both solutions. The differences between the dynamical Euler angles $\underset{1}{\psi}$, $\underset{1}{\theta}$, $\underset{1}{\varphi}$ (relating ITRS and DGRSC) in the Newtonian (N) and relativistic solutions demonstrate the influence of the indirect relativistic third–body perturbations. The dynamical Euler angles for both (Newtonian and relativistic) versions are converted by means of (10) into the kinematical Euler angles $\underset{0}{\psi}$, $\underset{0}{\theta}$, $\underset{0}{\varphi}$ (relating ITRS and KGRSC) also for the Newtonian (N) and relativistic solutions. The differences between the dynamical and kinematical Euler angles for the relativistic solution (evidently, within the post–Newtonian approximation the similar differences for the Newtonian version are practically the same) exposed in Appendix B (Tables (1)–(3)) improve the corresponding values given in (Bretagnon et al. 1997). The differences between the kinematical Euler angles in the Newtonian and relativistic solutions (Tables (4)–(6) of Appendix B) differ only slightly from the corresponding differences between the dynamical angles (this discrepancy reveals only in terms of the third and higher power of time). For the sake of completeness, we reproduce also the series for the geodesic rotation vector $\mathbf{F}_C$ (Tables (7)–(9) of Appendix B).

Let us note once again that the Newtonian and relativistic SMART solutions are distinguished just with respect to the employed luni–solar coordinates as stated above. When converting from DGRSC to KGRSC both these solutions are transformed practically in the same manner as prescribed by the geodesic rotation (10). Newtonian solutions in DGRSC and KGRSC are differ by relativistic terms caused by the mutual rotation of reference systems not affecting the Newtonian nature of the solution itself.

The final expansions show that the differences in the Euler angles for the Newtonian and relativistic solutions are of the order of 35 μas over 20 yrs (cf. the precision of SMART97 of 2 μas) and 150 μas over 100 yrs (cf. the precision of SMART97 of 12 μas). Therefore, the relativistic indirect third–body perturbations found here are within the accuracy of SMART97 theory and may be used to improve this theory.

5. OPEN QUESTIONS

Before concluding this paper we would like to mention some points of possible confusion due to the lack of rigorous astronomical definitions.

1. The main astronomical reference systems ICRS and GCRS being now well defined (IAU 2001) it is necessary that the orientation of the reference systems underlying the existing ephemerides such as DE, LE, EPM, VSOP, ELP, SMART, etc., be rigorously related with these fundamental RSs. The constant rotation matrices P_C, P_Q relating ICRS with BRSC and BRSQ, respectively, just illustrate the adjustment of the VSOP RS (BRSC) to ICRS.

2. The geodesic rotation vector $\mathbf{F}$ is given by its first order derivative (see, e.g., Bretagnon and Brumberg 2003) but no one IAU resolution specifies the arbitrary constant involved in integrating this equation. In our work we imply the condition $\mathbf{F} = 0$ for J2000.0 as suggested in (Bretagnon and Brumberg 2003).

3. In spite of the IAU definition of the epoch J2000.0 with respect to TT the existing planetery and lunar theories often make use of the epoch J2000.0 with respect to TDB.

4. The use of AS (astronomical system of units) in BRS and GRS needs to be specified as well. In using 'practical' time scales TDB (or else $\mathrm{T_{eph}}$) and TT instead of 'theoretical' time scales TCB and TCG one has to deal with the scaling factors $(1 - L_B)$ and $(1 - L_G)$ for the coordinates $\mathbf{x}$, $\mathbf{w}$ and mass–coefficients GM, $G\hat{M}$ in BRS and GRS, respectively (Brumberg et al. 1998). L_G is now fixed by the IAU Resolution B1 (2000) as a defining constant. By contrast, L_B and L_C related by means of $1 - L_B = (1 - L_C)(1 - L_G)$ depend on specific planetary theories. Within the currently employed approximation $M = \hat{M}$ one has (13). The AS unit of time $d = 1\,\mathrm{day} = 86400s$ is defined directly by its relationship to the SI unit of time. The AS unit of mass is defined as the mass of the Sun M_S. The AS unit of length AU is defined by the condition $k = \sqrt{G} = 0.01720209895$ provided that the units of measurement are d, M_S and AU. This definion involves the relationship with the SI unit of length

$$AU = \left(\frac{GM_S d^2}{k^2}\right)^{1/3} . \qquad (20)$$

in form $1\,AU = \chi$ m. The defining relation (20) results from the Kepler's third law in standard designations $n^2 a^3 = GM_S$. To distinguish between TDB and TCB values let us mark TDB values by the asterisk *. One has (in SI units)

$$a^* = (1 - L_B)a\,, \qquad v^* = v\,, \qquad (GM_S)^* = (1 - L_B)(GM_S), \qquad n^* = (1 - L_B)^{-1} n \qquad (21)$$

with the third Kepler's law $n^{*2} a^{*3} = (GM_S)^*$. In astronomical units there results

$$n^2 a^3 = GM_S\,\chi^3\,, \qquad n^{*2} a^{*3} = (GM_S)^*\,\chi^{*3} \qquad (22)$$

where $1(AU)_{\mathrm{TDB}} = \chi^*$ m. Hence,

$$\left(\frac{\chi^*}{\chi}\right)^3 = \left(\frac{n^*}{n}\right)^2 \left(\frac{a^*}{a}\right)^3 \frac{M_S}{M_S^*} . \qquad (23)$$

Therefore, the standard AS values $d = 1$ day, $M_S = 1$, $AU = 1$ correspond to BRS with TCB. When using BRS with TDB there results $d^* = (1 - L_B)d = (1 - L_B)$ day and one has to choose between two currently used options.

First option:

$$M_S^* = (1 - L_B)M_S = (1 - L_B)\,, \qquad AU^* = (1 - L_B)AU = (1 - L_B)\,, \qquad \chi^* = \chi \tag{24}$$

(the unit of mass $= (1 - L_B)$ solar mass of AS TCB, the unit of length $= (1 - L_B)$ astronomical units of length of AS TCB, all formulae (21) are valid both in SI and AS units, the same numerical value of the AS unit of length in m both for TDB and TCB).

Second option (based on E.M.Standish comments at the GA IAU 2003):

$$M_S^* = M_S = 1\,, \qquad AU^* = AU = 1\,, \qquad \chi^* = (1 - L_B)^{1/3}\,\chi \tag{25}$$

resulting to

$$\frac{a^*}{\chi^*} = (1 - L_B)^{2/3}\,\frac{a}{\chi}\,, \qquad \frac{v^*}{\chi^*} = (1 - L_B)^{-1/3}\,\frac{v}{\chi} \tag{26}$$

or just

$$a^* = (1 - L_B)^{2/3}\,a\ [\mathrm{AU}], \qquad v^* = (1 - L_B)^{-1/3}\,v\ [\mathrm{AU/day}] \tag{27}$$

(the unit of mass = 1 solar mass of AS TCB, the unit of length = 1 astronomical units of length of AS TCB, formulae (21) are valid only in SI units to be replaced by (27) in AS units, the same numerical value of the heliocentric constant GM_S in SI units both for TDB and TCB).

6. CONCLUSION

The relativistic indirect third–body perturbations considered in this paper contribute within 35μas accuracy in the Euler angles determining the Earth orientation parameters (relating ITRS with DGRSC or KGRSC). Therefore, they are indeed of practical importance for the SMART solution. Using the formalism of (Bretagnon and Brumberg 2003) it is possible to compute the rotation vector $\mathbf{A}$ of GCRS→ITRS transformation for the Newtonian and relativistic SMART solutions and to find explicitly the relativistic contributions in the components of this vector. Denoting the triplet of the ITRS spatial coordinates by $\mathbf{y}$ we may represent the GCRS→ITRS in form

$$\mathbf{y} = T\,\underset{0}{\mathbf{w}} \tag{28}$$

$$T = \underset{0}{\hat{P}}(u)P_C\,. \tag{29}$$

or

$$T = D_3(\underset{0}{\varphi})D_1(-\underset{0}{\theta})D_3(-\underset{0}{\psi})D_1(\varepsilon)D_3(\chi)\,. \tag{30}$$

Introducing the rotation vector A one may use the rotation formula

$$T = R(\mathbf{A}), \qquad R(\mathbf{A})\mathbf{x} = \mathbf{x} - \sin a(\hat{\mathbf{A}} \times \mathbf{x}) + (1 - \cos a)[\hat{\mathbf{A}} \times (\hat{\mathbf{A}} \times \mathbf{x})] \tag{31}$$

where $a = |\mathbf{A}|$ is the rotation angle, $\hat{\mathbf{A}} = \mathbf{A}/a$ is the unit vector along the rotation axis, $\mathbf{x}$ is an arbitrary coordinate vector. Vector A is given in (Bretagnon and Brumberg 2003) in three forms corresponding to 'dynamical' representation with three Euler angles, 'classical kinematical' representation (precession/nutation, diurnal rotation and polar motion) and modern 'kinematical' representation involving the non–rotating origin. The first representation in terms of $\underset{0}{\psi}$, $\underset{0}{\theta}$, $\underset{0}{\varphi}$ is most closely related with the SMART solution. Evaluating the variation $\delta\mathbf{A}$ between the relativistic and Newtonian values of $\mathbf{A}$ one may find the influence of the relativistic terms on the GCRS→ITRS transformation (see (A.24), (A.25) in Bretagnon and Brumberg 2003). This work is now in progress.

Numerical values exposed in this paper are given mainly for the illustration purposes. The complete expansions involved in the Newtonian and relativistic SMART solutions may be available in the electronic form by request to the second author .

7. REFERENCES

Bretagnon, P.: 2002, *Trans. Inst. Appl. Astron.* **8**, 33

Bretagnon, P. and Francou, G.: 1988, *Astron. Astrophys.* **202**, 309

Bretagnon, P. and Brumberg, V.A.: 2003, *Astron. Astrophys.* **408**, 387

Bretagnon, P., Rocher, P., and Simon, J.L.: 1997, *Astron. Astrophys.* **319**, 305

Bretagnon, P., Francou, G., Rocher, P., and Simon, J.L.: 1998,*Astron. Astrophys.* **329**, 329

Bretagnon, P., Fienga, A., and Simon, J.-L.: 2003, *Astron. Astrophys.* **400**, 785

Brumberg, V.A.: 1991, *Essential Relativistic Celestial Mechanics*, Hilger, Bristol

Brumberg, V.A.: 1996, In: N.Capitaine, B.Kolaczek, and S.Debarbat (eds.), *Earth Rotation, ReferenceSystems in Geodynamics and Solar System* (Journées 1995), Warsaw, 29

Brumberg, V.A.: 1997, In: I.M.Wytrzyszczak, J.H.Lieske, and R.A.Feldman (eds.), *Dynamics and Astrometryof Natural and Artificial Celestial Bodies*, Kluwer, 439

Brumberg, V.A.: 1998, *Trans. Inst. Appl. Astron.* **3**, 152 (in Russian)

Brumberg, V.A., Bretagnon, P. and Guinot, B.: 1996, *Celes. Mech. & Dyn. Astron.* **64**, 231

Brumberg, V.A., Bretagnon, P., Capitaine, N. et al.: 1998, In: J.Anderson (ed.), *Highlights of Astronomy*, **11A**, 194

IAU: 2001, *IAU Information Bull.* **88**, 28 (Errata: ibid., **89**, 4, 2001).

Klioner S.A. and Soffel M. 1998, In: J.Andersen (ed.)*Highlights of Astronomy* **11A**, 173

Klioner S.A. and Soffel M. 1999, In: J.Henrard and S.Ferraz–Mello (eds.), *Impact ofModern Dynamics in Astronomy*, Kluwer, 435

Mathews, P.M., Herring, T.A., and Buffet, B.A.: 2002, *J. Geophys. Res.* **107**, B4

APPENDIX

A. Fundamental arguments

As stated above, the expansions of the present paper have 12 trigonometrical arguments as follows:

$$\begin{aligned}
\lambda_1(t) &= 4.40260867435 + 26087.9031415742\,t\,, \\
\lambda_2(t) &= 3.17614652884 + 10213.2855462110\,t\,, \\
\lambda_3(t) &= 1.75347029148 + 6283.0758511455\,t\,, \\
\lambda_4(t) &= 6.20347594486 + 3340.6124266998\,t\,, \\
\lambda_5(t) &= 0.59954632934 + 529.6909650946\,t\,, \\
\lambda_6(t) &= 0.87401658845 + 213.2990954380\,t\,, \\
\lambda_7(t) &= 5.48129370354 + 74.7815985673\,t\,, \\
\lambda_8(t) &= 5.31188611871 + 38.1330356378\,t\,, \\
D(t) &= 5.19846640063 + 77713.7714481804\,t\,, \\
F(t) &= 1.62790513602 + 84334.6615717837\,t\,, \\
l(t) &= 2.35555563875 + 83286.9142477147\,t\,, \\
\phi(u) &= 4.89496121282 + 2301216.7536515365\,u\,.
\end{aligned}$$

The mean longitudes of eight major planets $\lambda_i(t)$ are referred to ICRS (to the reference system of of DE403 in practice). Therefore, their constant parts are those given in (Bretagnon et al. 1998). Their frequencies are also issued from (Bretagnon et al. 1998) but without taking into account the precession. The Delaunay arguments $D(t)$, $F(t)$, $l(t)$ of the lunar theory are taken from (Bretagnon et al. 1998). Originally, these arguments are functions of t =TDB but in accordance with (18) they are used here just as the linear functions of u =TT. The last argument, $\phi(u)$, representing the linear part of the Euler angle φ of the Earth's rotation is taken from (Bretagnon et al. 1998) as well. All values here and below are given using the astronomical unit as the unit of length (1 AU=149597870.691 km as in DE403) and 1000 Julian years (365250 Julian days) as the unit of time (tjy).

B. Final expansions

This Appendix contains the initial terms of the final series described in Section 4. All series are presented in form of (19). The data in the 6–column Tables 1–9 read: ordinal number of the term, components of the trigonometric argument (mean longitudes of eight major planets from Mercury to Neptune, arguments D, F, l of the lunar theory and the Earth's rotation angle ϕ) given to show the physical meaning of the term, coefficient X, the phase angle ψ of the argument, the frequency ν of the argument and exposant α of power of t. The time argument t is either TDB or TT as indicated explicitly. The negative components of the trigonometric arguments are underlined. The coefficients of the series in Tables 1–6 are given in radians. The coefficients of the series in Tables 7-9 for $\mathbf{F}_C$ are given in arcseconds to facilitate their comparison with our previous results and the results of other authors. The coefficient of the term No. 11 in Table 9 corresponds to geodesic precession. The truncation level is 0.1E-11 over 1000 yrs in Tables 1–3, 0.5E-11 over 100 yrs or 0.1E-9 over 1000 yrs in Tables 4–6, and 0.1E-7 arcsecond over 1000 yrs in Tables 7–9.

Table 1. Differences $\psi_1 - \psi_0$ (TT) (GRSC)

1	001000000000	.742300349 − 09	.466926087 + 01	.628307585 + 04	0
2	000000000000	.397739929 − 10	.000000000 + 00	.000000000 + 00	0
3	001000001$\underline{1}$00	.145708061 − 10	.252987892 + 01	.337814272 + 03	0
4	002000000000	.930524508 − 11	.462564700 + 01	.125661517 + 05	0
5	00002$\underline{5}$000000	.203878362 − 11	.709948865 + 00	.711354700 + 01	0
6	000000000000	.930785387 − 04	.000000000 + 00	.000000000 + 00	1
7	001000000000	.474945452 − 10	.260332598 + 01	.628307585 + 04	1
8	001000001$\underline{1}$00	.355965693 − 11	.960586737 + 00	.337814272 + 03	1
9	002000000000	.116661765 − 11	.259734787 + 01	.125661517 + 05	1
10	000000000000	.244280065 − 06	.314159265 + 01	.000000000 + 00	2
11	001000001$\underline{1}$00	.299152471 − 10	.417657693 + 01	.337814272 + 03	2
12	001000000000	.203316536 − 11	.956674751 + 00	.628307585 + 04	2
13	002000000000	.172295028 − 11	.340748453 + 01	.125661517 + 05	2
14	000000000000	.365594907 − 08	.314159265 + 01	.000000000 + 00	3
15	001000001$\underline{1}$00	.117386023 − 10	.261099272 + 01	.337814272 + 03	3
16	002000000000	.121017860 − 11	.500979962 + 01	.125661517 + 05	3
17	000000000000	.579411556 − 08	.000000000 + 00	.000000000 + 00	4
18	001000001$\underline{1}$00	.269183168 − 11	.106628445 + 01	.337814272 + 03	4
19	000000000000	.388062532 − 10	.314159265 + 01	.000000000 + 00	5
20	000000000000	.247036977 − 10	.314159265 + 01	.000000000 + 00	6

Table 2. Differences $\underset{1}{\theta} - \underset{0}{\theta}$ (TT) (GRSC)

1	00100000$\underline{1}$100	.631723382 − 11	.410068897 + 01	.337814272 + 03	0
2	000000000000	.322968763 − 11	.000000000 + 00	.000000000 + 00	0
3	00002$\underline{5}$000000	.957304138 − 12	.524343393 + 01	.711354700 + 01	0
4	000000000000	.473867892 − 10	.000000000 + 00	.000000000 + 00	1
5	00100000$\underline{1}$100	.154310985 − 11	.252997314 + 01	.337814272 + 03	1
6	000000000000	.947355383 − 08	.000000000 + 00	.000000000 + 00	2
7	00100000$\underline{1}$100	.883496788 − 11	.569290077 + 01	.337814272 + 03	2
8	000000000000	.228920555 − 07	.314159265 + 01	.000000000 + 00	3
9	00100000$\underline{1}$100	.216768803 − 11	.418750886 + 01	.337814272 + 03	3
10	000000000000	.653596505 − 10	.314159265 + 01	.000000000 + 00	4
11	000000000000	.184605964 − 09	.000000000 + 00	.000000000 + 00	5
12	000000000000	.179171613 − 11	.314159265 + 01	.000000000 + 00	6

Table 3. Differences $\underset{0}{\varphi} - \underset{1}{\varphi}$ (TT) (GRSC)

1	00100000$\underline{1}$100	.158810048 − 10	.567147813 + 01	.337814272 + 03	0
2	000000000000	.103746803 − 10	.314159265 + 01	.000000000 + 00	0
3	00002$\underline{5}$000000	.213211607 − 11	.376137398 + 01	.711354700 + 01	0
4	000000000000	.191527491 − 10	.000000000 + 00	.000000000 + 00	1
5	001000000000	.423492232 − 11	.466929724 + 01	.628307585 + 04	1
6	00100000$\underline{1}$100	.387872333 − 11	.409816916 + 01	.337814272 + 03	1
7	000000000000	.265537808 − 06	.000000000 + 00	.000000000 + 00	2
8	00100000$\underline{1}$100	.274735254 − 10	.104917664 + 01	.337814272 + 03	2
9	002000000000	.164479145 − 11	.278147098 + 00	.125661517 + 05	2
10	000000000000	.388849875 − 08	.000000000 + 00	.000000000 + 00	3
11	00100000$\underline{1}$100	.115340467 − 10	.575444761 + 01	.337814272 + 03	3
12	002000000000	.116412840 − 11	.185998408 + 01	.125661517 + 05	3
13	000000000000	.631104235 − 08	.314159265 + 01	.000000000 + 00	4
14	00100000$\underline{1}$100	.265593077 − 11	.420628407 + 01	.337814272 + 03	4
15	000000000000	.380544861 − 10	.000000000 + 00	.000000000 + 00	5
16	000000000000	.267916561 − 10	.000000000 + 00	.000000000 + 00	6

Table 4. Differences $\underset{0}{\psi_N} - \underset{0}{\psi}$ (TT) (KGRSC)

1	00 1 00000$\underline{1}$100	.231266586 − 10	.252922703 + 01	.337814272 + 03	0
2	0010$\underline{19}$03000000	.759550646 − 11	.284970036 + 01	.980309527 + 00	0
3	00 1 00000$\underline{1}$100	.780322332 − 08	.959162770 + 00	.337814272 + 03	1
4	00 2 000000000	.115291472 − 08	.365346519 + 00	.125661517 + 05	1
5	00 2 000002000	.199920367 − 09	.447910095 + 01	.167993695 + 06	1
6	00 2 00000$\underline{2}$200	.188795275 − 09	.191827564 + 01	.675628545 + 03	1
7	00 1 00000$\underline{1}$100	.190446025 − 08	.568655793 + 01	.337814272 + 03	2
8	00 2 000000000	.562966495 − 09	.194262028 + 01	.125661517 + 05	2
9	00 0 000000000	.117110967 − 09	.000000000 + 00	.000000000 + 00	2
10	00 0 000000000	.382641250 − 09	.314159265 + 01	.000000000 + 00	3
11	00 1 00000$\underline{1}$100	.234166521 − 09	.420126686 + 01	.337814272 + 03	3
12	00 2 000000000	.137527571 − 09	.352339649 + 01	.125661517 + 05	3
13	00 0 000000000	.250403635 − 08	.000000000 + 00	.000000000 + 00	4

Table 5. Differences $\underset{0}{\theta}_N - \underset{0}{\theta}$ (TT) (KGRSC)

1	001000001100	.123517565 − 10	.410001583 + 01	.337814272 + 03	0
2	001000001100	.416734462 − 08	.252995484 + 01	.337814272 + 03	1
3	002000000000	.499381069 − 09	.507773508 + 01	.125661517 + 05	1
4	001000001100	.101645481 − 08	.965285401 + 00	.337814272 + 03	2
5	000000000000	.323583375 − 09	.314159265 + 01	.000000000 + 00	2
6	002000000000	.243819609 − 09	.373329495 + 00	.125661517 + 05	2
7	000000000000	.859605900 − 08	.314159265 + 01	.000000000 + 00	3
8	001000001100	.124723404 − 09	.577801150 + 01	.337814272 + 03	3

Table 6. Differences $\underset{0}{\varphi} - \underset{0}{\varphi}_N$ (TT) (KGRSC)

1	00 1 000001100	.212112060 − 10	.567081957 + 01	.337814272 + 03	0
2	00101903000000	.265955171 − 11	.625381360 + 00	.980309527 + 00	0
3	00 1 000001100	.715812097 − 08	.410075543 + 01	.337814272 + 03	1
4	00 2 000000000	.105778263 − 08	.350693917 + 01	.125661517 + 05	1
5	00 2 000002000	.183423115 − 09	.133750830 + 01	.167993695 + 06	1
6	00 2 000002200	.173274092 − 09	.505986832 + 01	.675628545 + 03	1
7	00 1 000001100	.174700634 − 08	.254497058 + 01	.337814272 + 03	2
8	00 2 000000000	.516512839 − 09	.508421183 + 01	.125661517 + 05	2
9	00 0 000000000	.107461963 − 09	.314159265 + 01	.000000000 + 00	2
10	00 0 000000000	.340588092 − 09	.000000000 + 00	.000000000 + 00	3
11	00 1 000001100	.214853535 − 09	.105831374 + 01	.337814272 + 03	3
12	00 2 000000000	.126190931 − 09	.381968078 + 00	.125661517 + 05	3
13	00 0 000000000	.250626807 − 08	.314159265 + 01	.000000000 + 00	4

Table 7. $c^{-2}F^1_C$(TT) (GRSC) (coefficients in arcseconds)

1	00 1000001100	.130302301 − 05	.410074370 + 01	.337814272 + 03	0
2	00 0000000000	.666116309 − 06	.000000000 + 00	.000000000 + 00	0
3	00 0025000000	.197458153 − 06	.524343393 + 01	.711354700 + 01	0
4	0813000000000	.335755401 − 07	.214693239 + 01	.262983048 + 02	0
5	03 5000000000	.274317722 − 07	.418857071 + 01	.775522617 + 03	0
6	00 1000000100	.213556899 − 07	.338135431 + 01	.906177374 + 05	0
7	00 1000000100	.155700142 − 07	.615764243 + 01	.780515857 + 05	0
8	00 0020000000	.113693032 − 07	.572693663 + 01	.105938193 + 04	0
9	00 0000000000	.956621901 − 05	.000000000 + 00	.000000000 + 00	1
10	00 1000000000	.311750820 − 07	.466912341 + 01	.628307585 + 04	1
11	00 0000000000	.195442531 − 02	.000000000 + 00	.000000000 + 00	2
12	00 1000000000	.137268118 − 07	.454103182 + 01	.628307585 + 04	2
13	00 0000000000	.601796456 − 03	.000000000 + 00	.000000000 + 00	3
14	10 0000000000	.520765790 − 05	.314159265 + 01	.000000000 + 00	4
15	10 0000000000	.192780654 − 06	.314159265 + 01	.000000000 + 00	5

Table 8. $c^{-2}F_C^2(\mathrm{TT})$ (GRSC) (coefficients in arcseconds)

1	00 1000001100	.130305083 − 05	.567154555 + 01	.337814272 + 03	0
2	00 0000000000	.851215459 − 06	.314159265 + 01	.000000000 + 00	0
3	00 0025000000	.174934562 − 06	.376137398 + 01	.711354700 + 01	0
4	0813000000000	.326128037 − 07	.373005969 + 01	.262983048 + 02	0
5	03 5000000000	.267920049 − 07	.260932664 + 01	.775522617 + 03	0
6	00 1000000100	.213538259 − 07	.181060019 + 01	.906177374 + 05	0
7	00 1000000100	.155685742 − 07	.144520895 + 01	.780515857 + 05	0
8	00 0020000000	.114422842 − 07	.415710213 + 01	.105938193 + 04	0
9	00 0000000000	.140865733 − 05	.000000000 + 00	.000000000 + 00	1
10	00 1000000000	.347458181 − 06	.466925775 + 01	.628307585 + 04	1
11	00 0025000000	.129546311 − 07	.461641251 + 01	.711354700 + 01	1
12	00 0000000000	.217843218 − 01	.000000000 + 00	.000000000 + 00	2
13	00 1000000000	.232329482 − 07	.252863427 + 01	.628307585 + 04	2
14	00 0000000000	.158568418 − 03	.314159265 + 01	.000000000 + 00	3
15	10 0000000000	.121552283 − 04	.314159265 + 01	.000000000 + 00	4
16	10 0000000000	.105689498 − 06	.000000000 + 00	.000000000 + 00	5

Table 9. $c^{-2}F_C^3(\mathrm{TT})$ (GRSC) (coefficients in arcseconds)

1	001000000000	.153110555 − 03	.466926126 + 01	.628307585 + 04	0
2	000000000000	.624062608 − 05	.000000000 + 00	.000000000 + 00	0
3	002000000000	.191850004 − 05	.462609811 + 01	.125661517 + 05	0
4	000000001000	.371707163 − 06	.486077422 + 00	.777137714 + 05	0
5	004830000000	.174492597 − 06	.283203731 + 01	.352311373 + 01	0
6	001010000000	.207986602 − 06	.585127088 + 01	.575338489 + 04	0
7	022000000000	.127215510 − 06	.441820021 + 01	.786041939 + 04	0
8	011000000000	.912767135 − 07	.613529387 + 01	.393020970 + 04	0
9	001001000000	.788781114 − 07	.559096259 + 01	.606977676 + 04	0
10	001200000000	.531395177 − 07	.358258491 + 01	.398149002 + 03	0
11	000000000000	.191988304 + 02	.000000000 + 00	.000000000 + 00	1
12	001000000000	.944199242 − 05	.267808013 + 01	.628307585 + 04	1
13	002000000000	.236615606 − 06	.263508345 + 01	.125661517 + 05	1
14	004830000000	.222412262 − 07	.153676866 + 01	.352311373 + 01	1
15	000025000000	.113644375 − 07	.884980375 − 01	.711354700 + 01	1
16	000000000000	.134870204 − 03	.314159265 + 01	.000000000 + 00	2
17	001000000000	.399915448 − 06	.107231782 + 01	.628307585 + 04	2
18	002000000000	.170005184 − 07	.867218932 + 00	.125661517 + 05	2
19	000000000000	.181895781 − 04	.314159265 + 01	.000000000 + 00	3
20	001000000000	.132810927 − 07	.584295983 + 01	.628307585 + 04	3
21	100000000000	.252135692 − 06	.000000000 + 00	.000000000 + 00	4