THE BCRS AND THE LARGE SCALE STRUCTURE OF THE UNIVERSE

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ABSTRACT. The BCRS is presently defined for an isolated solar system by ignoring effects from cosmology. Various problems that arise if one tries to match the BCRS with a cosmological metric that describes the expansion of the universe are discussed. An approximate solution for the BCRS with cosmological constant Λ is given.

1. THE BCRS AND COSMOLOGY

According to IAU-Resolution B1.3 the BCRS with coordinates $(t, \mathbf{x})$ is defined by a metric tensor of the form

$$\begin{aligned} g_{00} &= -1 + \frac{2w}{c^2} - \frac{2w^2}{c^4} + \mathcal{O}(c^{-5}) \\ g_{0i} &= -\frac{4}{c^3} w^i + \mathcal{O}(c^{-5}) \\ g_{ij} &= \delta_{ij}\left(1 + \frac{2}{c^2} w\right) + \mathcal{O}(c^{-4}) \,. \end{aligned} \tag{1}$$

Here, the gravito-electric potential w generalizes the usual Newtonian gravitational potential U and the gravito-magnetic potential w^i describes gravitational effects resulting from moving gravitational sources. The order symbols in Eq.(1) indicates that the validity of the metric tensor is restricted to the first post-Newtonian approximation of Einstein's theory of gravity. In that approximation, using the harmonic gauge condition, the field equations read

$$\begin{aligned} \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) w &= -4\pi G \sigma + \mathcal{O}(c^{-4}) \,, \\ \nabla^2 w^i &= -4\pi G \sigma^i + \mathcal{O}(c^{-2}) \,. \end{aligned} \tag{2}$$

Here σ and σ^i are the gravitational mass and mass current density, respectively. Mathematically they are related to the energy-momentum tensor $T^{\mu\nu}$ by

$$\sigma = \frac{1}{c^2}\left(T^{00} + T^{ss}\right) \,, \quad \sigma^i = \frac{1}{c} T^{0i} \,. \tag{3}$$

For the present definition of the BCRS spacetime is assumed to be asymptotically flat, i.e.,

$$\lim_{\substack{r \to \infty \\ t=\mathrm{const}}} g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)\,. \tag{4}$$

This condition assumes that all kinds of gravitational sources outside the solar system, as well as the vacuum energy (or cosmological constant) that pervades every volume element of our universe, are ignored.

As is well known the (averaged) global mass-energy distribution on large cosmic scales determines the global geometry of the universe as well as its global dynamical development. For a description of the universe on large scales the so-called "cosmological principle" is of great value. This principle says that on very large scales our universe is homogeneous and isotropic. Mathematically the Robertson-Walker metric follows from this principle. For a flat space in suitable coordinates $(T, \mathbf{X})$ this metric takes the form

$$ds^2 = -c^2\, dT^2 + a^2(T)\, d\mathbf{X}^2\,, \tag{5}$$

where $a(T)$ is the cosmic scale factor. Such a cosmic metric has profound consequences for astrometry: it describes cosmic red shift effects due to the expansion of the universe and leads to various distance measures like parallax distance, luminosity distance, angular diameter distance or proper motion distance that differ from each other (Weinberg, 1972). The question of empirical validity of the cosmological principle has been investigated in detail in recent years (Lahav, 2000 and references quoted therein). Up to scales of order of some 100 Mpc the universe is obviously very clumpy and dominated by a hierarchical structure (our galaxy (0.03 Mpc), the local group (1-3 Mpc), the local supercluster (20-30 Mpc)). Deep red shift surveys like the 2dF Galaxy Redshift Survey show distinct structures like the great wall with dimension of $150 \times 70 \times 5$ Mpc. On scales larger than about 100 Mpc the cosmological principle, however, seems to be satisfied well. Further support comes from studies of the anisotropies of the Cosmic Microwave Background Radiation (CMBR), especially the data from the Wilkinson Microwave Anisotropy Probe WMAP. The data show (Lahav, 2000) that for scales larger than about $1000/H$ Mpc (H is the the Hubble constant in units of 100 km/s/Mpc) density fluctuations $\delta\rho/\rho$ are smaller than 10^{-4}. Thus the Robertson-Walker metric can be justified empirically for such large spatial scales. Theoretically one expects such a metric to result from some spatial averaging procedure (unfortunately such a rigorous averaging algorithm has not yet been worked out for Einstein's theory of gravity) and the question is what kind of signatures of this cosmic metric can be found locally, e.g., on solar-system scales.

The WMAP data just mentioned also contributes significantly to the present cosmological standard model. According to that model the age of our universe is about 13.7 billion years, the Hubble constant $H_0 = (71 \pm 4)$ km/s/Mpc and the total density parameter Ω that determines the global geometry of the universe is about 1, i.e. our universe is practically flat (which is also implied theoretically from the inflation scenario). More specifically the contribution to Ω from luminous matter is about 0.04, from dark matter 0.23 and from the vacuum energy 0.73.

2. RELATING THE BCRS WITH A COSMOLOGICAL METRIC

Due to the hierarchical structure of our cosmic neighbourhood one might extend the hierarchy of astronomical reference systems from the GCRS and the BCRS to some "GaCRS" (galactic celestial reference system), some "LoGrCRS" (local group celestial reference system) etc. In such a hierarchy each system will contain tidal forces due to effects from the external matter. Nevertheless, at a certain scale cosmologiocal effects can no longer be ignored and the expansion of the universe has to be taken into account.

Theoretically one faces the problem how to match a local metric such as the one defining the BCRS with the Robertson-Walker metric that describes the gravitational physics on large cosmic scales. As a first step in the present paper we considered the vacuum energy only. The vacuum energy can be described by a cosmological constant Λ. We started with the following ansatz for the local metric

$$
\begin{aligned}
g_{00} &= -1 + \frac{2}{c^2}\, w(t,\mathbf{x}) - \frac{2}{c^4}\, w^2(t,\mathbf{x}) + \mathcal{O}(c^{-5}), \\
g_{0i} &= -\frac{4}{c^3}\, w^i(t,\mathbf{x}) + \mathcal{O}(c^{-5}), \\
g_{ij} &= \delta_{ij}\left(1 + \frac{2}{c^2}\, w'(t,\mathbf{x})\right) + \mathcal{O}(c^{-4}).
\end{aligned} \tag{6}
$$

Since the constant Λ is very small ($\Lambda \approx 2 \cdot 10^{-52}\ \mathrm{m}^{-2}$) we neglect all terms $\mathcal{O}(\Lambda\, G)$ (G being the Newtonian gravitational constant). In this approximation and implying the gauge condition $w_{,t} + w^i{}_{,i} = \mathcal{O}(c^{-2})$ the field equations read

$$
w_{,ii} - \frac{1}{c^2} w_{,tt} = -4\,\pi\, G\,\sigma + c^2\,\Lambda, \tag{7}
$$

$$
w^i{}_{,jj} = -4\,\pi\, G\,\sigma^i. \tag{8}
$$

Denoting w_σ a potential satisfying

$$
w_{\sigma\ ,ii} - \frac{1}{c^2} w_{\sigma,tt} = -4\,\pi\, G\,\sigma, \tag{9}
$$

one gets

$$
w = w_\sigma + \frac{1}{6}\, c^2\,\Lambda\, x^i\, x^i, \tag{10}
$$

$$
w' = w_\sigma - \frac{1}{12}\, c^2\,\Lambda\, x^i\, x^i. \tag{11}
$$

Note that for positive values of Λ the vacuum yields a repulsive gravitational force. Potential w_σ denotes the contribution of ordinary solar system matter and the remaining terms describe the influence of the overall vacuum energy in the universe. In our local coordinates at this level of approximation the cosmological constant leads to a static cosmic tidal-like term that grows quadratically with coordinate distance r ($r^2 = x^i\, x^i$). Looking at orders of magnitude one finds that these cosmic tidal terms are completely negligible in the solar system (see also Cooperstock et al., 1998). Only at cosmic distances where the cosmic red shift are not negligible they play a role.

It is important to note that there exists a well-known exact solution of the Einstein field equations with cosmological term called the Schwarzschild-de Sitter solution:

$$
ds^2 = -Ac^2 dT^2 + A^{-1} d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{12}
$$

$$
A = 1 - \frac{2m}{\rho} - \frac{1}{3}\,\Lambda\,\rho^2, \tag{13}
$$

where m is the Schwarzschild radius (normalized mass) of a spherically symmetric body embedded in the de Sitter universe with cosmological constant Λ. In the limit $\Lambda = 0$ this solution coincides with the Schwarzschild solution in the Schwarzschild standard coordinates. In the limit $m = 0$ Eq. (12) describes the de Sitter cosmological solution (empty static universe with cosmological constant Λ). This well-known metric can be transformed into isotropic form which, neglecting terms of order $\mathcal{O}(m\,\Lambda)$, coincides with the metric (6) potentials with (10) for the case of a spherically symmetric mass distribution. On the other hand the same metric (12)–(13) can be transformed into the form (Robertson, 1928)

$$
\begin{aligned}
ds^2 &= g_{00}\,c^2\,dT^2 + g_{RR}\,d\mathbf{X}^2, && (14)\\
g_{00} &= -\left(\frac{1-\frac{m}{2\,a_\Lambda\,R}}{1+\frac{m}{2\,a_\Lambda\,R}}\right)^2, && (15)\\
g_{RR} &= a_\Lambda^2(T)\left(1+\frac{m}{2\,a_\Lambda\,R}\right)^4, && (16)\\
a_\Lambda(T) &= \exp\left(\sqrt{\frac{\Lambda}{3}}\,c\,T\right), && (17)
\end{aligned}
$$

which can be though of as a perturbed Fermi-Walker solution (5). This form of the metric can be used to match the BCRS metric (6) with (10) to the Robertson-Walker metric. Indeed, at the cosmic distances where the Λ terms in (10) play a role, the non-spherical part of the local potential w_σ and the local vector potential w^i can be neglected, and the rest can be directly matched to (14)–(17). This matching and further details will be published elsewhere.

The general Robertson-Walker metric (5) with arbitrary $a(T)$ can be also transformed in what we can call "local coordinates". Indeed, the transformation

$$
\begin{aligned}
t &= T + \frac{1}{2c^2}\,a\,\dot{a}^2\,R^2 + \mathcal{O}(R^4), && (18)\\
\mathbf{x} &= a(t)\,\mathbf{X}\left(1+\frac{1}{4\,c^2}\,\dot{a}^2\,R^2 + \mathcal{O}(R^4)\right) && (19)
\end{aligned}
$$

brings the metric (5) into the form

$$
ds^2 = \left(-1 + \frac{1}{c^2}\,\frac{\ddot{a}}{a}\,R^2 + \mathcal{O}(R^4)\right)\,c^2\,dt^2 + \left(1+\frac{1}{2\,c^2}\,\left(\frac{\dot{a}}{a}\right)^2\,R^2 + \mathcal{O}(R^4)\right)\,d\mathbf{x}^2. \qquad (20)
$$

The same procedure can be done to any order of R. In the limit of de Sitter universe the function $a(T)$ is defined by (17) and (20) coincides with the Schwarzschild-de Sitter metric and with the BCRS metric with Λ in the corresponding limits. The fact that local coordinates exist also for general Robertson-Walker metric fosters the hope that the BCRS can be also matched to the Robertson-Walker universe in the general case. Details will be published elsewhere.

We might finally ask about the contribution of visible and dark matter to the spacetime metric and formulate the 'Local Expansion Hypothesis': the cosmic expansion induced by ordinary (visible and dark) matter occurs on all length scales, i.e., also locally. The question whether this hypothesis is true or not has a long history and one must confess that this fundamental problem so far has no satisfying solution. If we forget about the Λ-term the famous Einstein-Strauss solution (Einstein, Strauss, 1945, Bonnor, 2000), where a pure static Schwarzschild solution without mass-energy inside of some spherical vacuole can be matched exactly to a global Robertson-Walker metric indicates that this hypothesis might be wrong. It has been argued that such a solution is unstable and cannot be generalized to situations of less symmetry but nevertheless the sign of warning is clear. This also indicates that the transition from the local metric to the cosmic one cannot be studied by looking at a single spherically symmetric density inhomogeneity; one rather has to study theoretically a basically clumpy universe, defining some suitable spatial averaging procedure and then study the limit of larger and larger scales. Likely the inhomogeneity scale determined by the two-point correlation function will also theoretically indicate the validity of the cosmic Robertson-walker metric.

In conclusion let us summarize that if one is interested in cosmology, radial coordinates of remote objects (e.g., quasars) should be defined with respect to a metric which turns into a cosmological metric (e.g. the Robertson-Walker one) in the limit of very large barycentric distances. Several possibilities to construct such a metric have been sketched above. The implications of such a "local cosmological" metric on the processing of astrometrical data should be further investigated.

3. REFERENCES

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