STABILITY OF EQUATORIAL SATELLITE ORBITS

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ABSTRACT. We study satellite orbits lying in the equatorial plane of a planet via the geometric methods of the theory of dynamical systems. To model the planetary gravitational potential, we expand it to the sixth zonal harmonic. The motion equations are regularized by means of McGehee-type transformations of the second kind. Naturally considering the motion to be collisionless and escapeless, we take into account the whole interplay among field parameters, total-energy level and angular momentum. This gives rise to various phase-portraits. In the most general case as regards the changes of sign of parameters, we meet: saddles generating simple or double homoclinic loops, double loops inside one loop of a larger double loop, centers surrounded by periodic and quasiperiodic trajectories, heteroclinic orbits, etc. Of course, less general cases lead to simpler phase portraits. Every type of phase orbit is translated in terms of physical motion. Such qualitative results are useful to the analysis of circumplanetary motion of major or infinitesimal satellites, rings, etc.

1. INTRODUCTION

Consider a planet that presents mass-distribution symmetry with respect to an axis. The equatorial motion of a satellite in the gravitational field of such a planet will be governed by the potential

$$U(\mathbf{q}) = \sum_{n=1}^{7} a_n / |\mathbf{q}|^n ,$$

where $\mathbf{q}$ is the radius vector of the satellite with respect to the mass centre of the planet and a_n are real parameters. We consider $a_1 > 0$ (the Newtonian term), $a_2 = 0$, $a_3 < 0$, as in the general planetary case in the solar system. To obtain the most general situations, we considered the whole sign interplay among $a_4 - a_7$.

Since we deal with satellites, we naturally consider the motion to be free of collision and escape. Collisionless dynamics is possible only for $a_7 < 0$, while escapeless dynamics is possible only for negative energy levels. To regularize the motion equations, we use the coordinates introduced by McGehee (1974). To identify the stability zones, we resort to the reduced 2D phase space, describing all possible phase curves, and using a foliation by the constant angular momentum. The stable orbits are either circular (relative equilibria) or noncircular (periodic and quasiperiodic); see Figure 1. The initial data that lead to quasiperiodic trajectories have positive Lebesgue measure.

2. BASIC EQUATIONS

Modelling the motion of a satellite in the considered field, the associated two-body problem can be reduced to a central-force problem. The planar motion of the satellite with respect to the planet is described by the Hamiltonian $H(\mathbf{q},\mathbf{p}) = |\mathbf{p}|^2/2 - \sum_{n=1}^{7} a_n/|\mathbf{q}|^n$, where $\mathbf{q} = (q_1,q_2)\in\mathbf{R}^2\backslash\{(0,0)\}$, $\mathbf{p} = (p_1,p_2) \in \mathbf{R}^2$ are the configuration vector and the momentum vector of the satellite, respectively. The problem admits the first integrals of angular momentum ($q_1p_2 - q_2p_1 = L = \text{constant}$) and of energy ($H(\mathbf{q},\mathbf{p}) = h/2 = \text{constant}$).

To regularize the motion equations, we apply the following sequence of McGehee-type transformations (McGehee 1974; see also Mioc and Stavinschi 2001, 2002):

$$\begin{aligned} r &= |\mathbf{q}|, \\ \theta &= \arctan(q_2/q_1), \\ \xi &= \dot{r} = (q_1p_1 + q_2p_2)/|\mathbf{q}|, \\ \eta &= r\dot{\theta} = (q_1p_2 - q_2p_1)/|\mathbf{q}|, \end{aligned} \tag{1}$$

which introduce standard polar coordinates,

$$\begin{aligned} x &= r^{7/2}\xi, \\ y &= r^{7/2}\eta, \end{aligned} \tag{2}$$

which scale down the velocity components, and a Sundman-type rescaling of time $d\tau = r^{-\alpha}dt$, $\alpha \in \mathbf{N}$. (This last transformation does not interest us in what follows.)

In this way we obtain regular equations of motion. Under the transformations (1)–(2), the angular momentum integral and the energy integral become respectively

$$y = Lr^{5/2}, \tag{3}$$

$$x^2 + y^2 = hr^7 + 2\sum_{n=1}^{7} a_n r^{7-n}, \tag{4}$$

whereas the singularity at $r = 0$ was replaced by the collision manifold $M_0 = \{(r,\theta,x,y) \mid r = 0, \theta \in S^1, x^2 + y^2 = 2a_7\}$ pasted on the phase space.

One sees that, for $a_7 < 0$, the motion is collisionless (see the expression of M_0). Also, by (4), that escape ($r \to \infty$) is not possible for $h < 0$. We shall hence study the motion within the framework of these two assumptions.

One could ask: why using McGehee-type coordinates to collisionless motion? On the one hand, the dynamics in these coordinates appears very simple. On the other hand, they will serve as a basis for a next investigations including collision and escape.

The regularized equations of motion do not contain θ explicitly, so we can factorize the flow by S^1. Next, we eliminate y between (3) and (4). In this way the phase-space dimension was reduced from 4 to 2. The energy integral in the (r,x)-plane will read

$$x^2 = f(r) = hr^7 + 2a_1r^6 - L^2r^5 + 2a_3r^4 + 2a_4r^3 + 2a_5r^2 + 2a_6r + 2a_7,$$

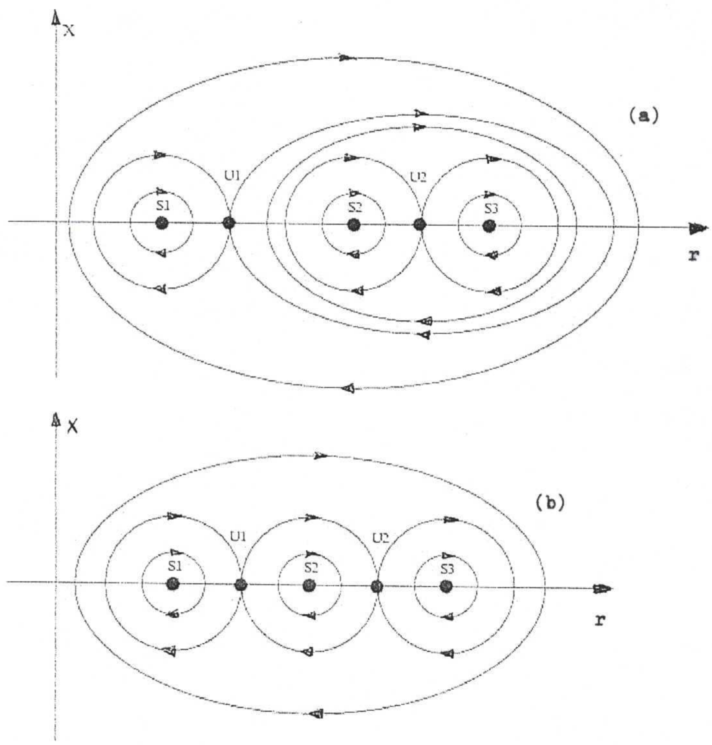
where we took into account the fact that $a_2 = 0$.

We shall describe the phase-space structure for negative energy levels, analyzing the behaviour of the function $x = \pm\sqrt{f(r)}$. Since $f(r) = x^2$, it must be nonnegative, so we have to consider only the regions on the real line where this condition is fulfilled. Also since r is a distance, we consider only the positive roots of $f(r)$. We also shall consider, for the same purpose, the positive roots of the polynomial $\tilde{f}(r) = df(r)/dr = 7hr^6 + 12a_1r^5 - 5L^2r^4 + 8a_3r^3 + 6a_4r^2 + 4a_5r + 2a_6$.

To deal with the most general case, we suppose that $f(r)$ has six changes of sign (the maximum possible) for $h < 0$, $a_1 > 0$, $a_3 < 0$, $a_7 < 0$. This entails a maximum of six positive roots (according to Descartes' rule) for $f(r)$, and five positive roots for $\tilde{f}(r)$.

3. PHASE-SPACE STRUCTURE

The phase-space structure for the considered case is plotted in Figure 1.



There exists a critical energy level h_c that creates, along with the interplay of the field parameters, three different phase portraits. The foliation performed by making $|L|$ increase points out a great variety of phase orbits, as well as bifurcations (corresponding to critical values of L) concretized by relative equilibria: three centres S (stable circular orbits) and two saddles U (unstable circular orbits).

Figure 1a plots the case $h < h_c < 0$. The inner saddle U_1 generates a double homoclinic loop, whose outer "petal" shelters another double homoclinic loop, generated by the outer saddle U_2.

There are three kinds of quasiperiodic and periodic orbits. The ones that correspond to phase curves surrounding all equilibria have significant eccentricities. The ones that correspond to phase curves lying inside the outer loop generated by U_1, but outside the double loop of U_2, have smaller eccentricities. By comparison, the quasiperiodic and periodic orbits situated inside the double loop of U_2 have smaller eccentricities, but we can say nothing about the eccentricities of such orbits lying inside the inner loop of U_1. All these orbits are stable.

The figure corresponding to the case $h_c < h < 0$ is wholly similar to Figure 1a, but inverted left-right.

Figure 1b plots the case $h = h_c$. The two saddles U_1 and U_2 generate: an inner homoclinic loop (U_1), an outer homoclinic loop (U_2), and two heteroclinic trajectories that link U_1 and U_2. There are two kinds of quasiperiodic and periodic orbits. The ones that correspond to phase curves surrounding all equilibria have significant eccentricities. The ones that correspond to phase curves lying inside the separatrices have smaller eccentricities. All these orbits are stable.

4. CONCLUDING REMARKS

To search for stability regions in our problem, we described the phase portraits for the whole interplay among field parameters (with the restrictions $a_1 > 0$, $a_2 = 0$, $a_3 < 0$, $a_7 < 0$), energy level (restricted to $h < 0$), and angular momentum.

The most important features of the model are:

4.1. There exist cases ($h < h_c$, $h > h_c$) in which two double homoclinic loops (one sheltered in one "petal" of another) associated to two saddles create five zones of stable quasiperiodic and periodic orbits (Figure 1a). There also exist cases ($h = h_c$) in which there are four zones of stable quasiperiodic and periodic orbits (Figure 1b).

4.2. The sets of quasiperiodic and noncircular periodic orbits have positive Lebesgue measure. Indeed, choosing initial data on such an orbit, and considering the foliations performed, in a neighbourhood of this point there exists an open set of initial data that lead to the same kind of orbit.

4.3. The role of the angular momentum is of the same importance as that of the energy. It creates bifurcations within the same energy level.

4.4. Our results were obtained for the maximum number of changes of sign of $f(r)$ and $\tilde{f}(r)$. Simpler mathematical situations entail simpler phase portraits.

4.5. Such qualitative are useful to analyze the stability of the circumplanetary motion of major or infinitesimal satellites, rings, etc.

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5. REFERENCES

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