NEW FORMULAE OF RELATIONS AMONG UT1, GAST, and ERA

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ABSTRACT. By using the latest precession theory of Fukushima and the nutation theory of Shirai and Fukushima, we numerically obtained a new approximation of the quantity $s + XY/2$ and the coefficients of the linear relation between the Earth Rotation Angle and UT1. These are key formulae in the post-2003 IAU formulation to connect the International Celestial Reference Frame and the International Terrestrial Reference Frame. Also we modified the pre-2003 IAU formula converting Greenwich Apparent Sidereal Time (GAST) from/to UT1 so as to be compatible with the modern observation of Earth orientation. The difference from the computation of Capitaine et al. is significant only in the secular components. This is due to the difference in the adopted precession formulae. From the viewpoint of approximation, we prefer the approach of GAST-UT1 relation since it requires the fewer terms in achieving the same degree of approximation.

1. INTRODUCTION

One of the most important procedures in the fundamental astronomy is the transformation between the two representative reference systems, the International Celestial Reference System (ICRS) and the International Terrestrial Reference System (ITRS). At its XXIVth General Assembly at Manchester, the International Astronomical Union (IAU) changed the policy to conduct the transformation. Since the new approach is to be used from 2003, we call the new and the old formulations the post- and the pre-2003 IAU formulations throughout this article.

The precession, the nutation, and the sidereal rotation are tightly connected with each other so that a change of any part of them inevitably induces the appropriate alterations of the others. Recently we developed a new formulation of precession (Fukushima 2003) such that its combination with our previous work on the nutation theory (Shirai and Fukushima 2001) provides a satisfactory agreement with the VLBI observation of celestial pole offsets. Its most significant feature is that not only the lunisolar precession formula but also the planetary precession one is improved. Actually the latter is compatible with the latest planetary/lunar ephemeris, DE405. In the light of the above statement, however, it is far from the completion to update only the theories of precession and nutation, which specify the true equator and equinox of date. Rather requested is a suitable replacement of the procedure of sidereal rotation such that the whole new formulation as a package is compatible with the modern observation of Earth orientations. In this short paper, we report our trials for both the pre- and post-2003 formulations using our new precession and nutation theories.

2. ANOTHER ANGLE TO SPECIFY CEO, q

As Aoki and Kinoshita (1983) clearly stated, Newcomb's definition of UT1 is based on the usage of *departure point* in specifying the longitude origin along the true equator of date. This is no other than the CEO, the Non-Rotating Origin (NRO) on the true equator. The angle s has been mainly used in locating the CEO. On the other hand, Aoki and Kinoshita (1983) provided the analytical expression of q, the angle between the true equinox and the CEO after the correction of the so-called nutation in RA as simply as $(\Delta q - \Delta\psi\cos\epsilon_{\mathrm{A}})_{\mathrm{A}} = -3.88t + 2.64\sin\Omega + 0.063\sin 2\Omega$ where the unit is mas. The absence of high order trends and mixed secular terms indicates a promising direction. In the below, we develop a new approach to evaluate the angle difference.

The rotational operation converting X, the x-axis of ICRF, to the CEO is expressed by the following five rotation matrices;

$$\mathcal{R}_3(q)(\mathcal{NP})_{\mathrm{NEW}} = \mathcal{R}_3(q)\mathcal{R}_1(-\epsilon)\mathcal{R}_3(-\psi)\mathcal{R}_1(\overline{\varphi})\mathcal{R}_3(\overline{\gamma}). \tag{1}$$

Then, the angular velocity vector of the CEO with respect to an inertial reference frame is expressed as

$$\omega_{\mathrm{CEO}} = \left(\frac{\mathrm{d}q}{\mathrm{dt}}\right)\vec{e}_{\mathrm{P}} - \left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)\vec{e}_{\mathrm{C}} - \left(\frac{\mathrm{d}\epsilon}{\mathrm{dt}}\right)\vec{e}_{\mathrm{Q}} + \left(\frac{\mathrm{d}\overline{\varphi}}{\mathrm{dt}}\right)\vec{e}_{\mathrm{N}} + \left(\frac{\mathrm{d}\overline{\gamma}}{\mathrm{dt}}\right)\vec{e}_{\mathrm{Z}}, \tag{2}$$

where $\vec{e}_{\mathrm{A}}$ means the unit vector toward a point A. The points additionally used are Z as the ICRF pole, C as the ecliptic pole of date, and P as the equatorial pole of date.

Since the CEO is a non-rotating origin, its inertial motion has no rotational component around P. This is expressed in terms of the angular velocity vector as $\omega_{\mathrm{CEO}} \cdot \vec{e}_{\mathrm{P}} = 0$. By solving this equation, we derive the differential equation of q as

$$\frac{\mathrm{d}q}{\mathrm{dt}} = \frac{\mathrm{d}\psi}{\mathrm{dt}}\left(\vec{e}_{\mathrm{P}}\cdot\vec{e}_{\mathrm{C}}\right) + \frac{\mathrm{d}\epsilon}{\mathrm{dt}}\left(\vec{e}_{\mathrm{P}}\cdot\vec{e}_{\mathrm{Q}}\right) - \frac{\mathrm{d}\overline{\varphi}}{\mathrm{dt}}\left(\vec{e}_{\mathrm{P}}\cdot\vec{e}_{\mathrm{N}}\right) - \frac{\mathrm{d}\overline{\gamma}}{\mathrm{dt}}\left(\vec{e}_{\mathrm{P}}\cdot\vec{e}_{\mathrm{Z}}\right). \tag{3}$$

Noting the fact that $\vec{e}_{\mathrm{P}}\cdot\vec{e}_{\mathrm{Q}} = 0$ and using the expressions

$$\vec{e}_{\mathrm{P}} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \qquad \vec{e}_{\mathrm{N}} = \begin{pmatrix} \cos\overline{\gamma} \\ \sin\overline{\gamma} \\ 0 \end{pmatrix}, \qquad \vec{e}_{\mathrm{C}} = \begin{pmatrix} \sin\overline{\varphi}\sin\overline{\gamma} \\ -\sin\overline{\varphi}\cos\overline{\gamma} \\ \cos\overline{\varphi} \end{pmatrix}, \tag{4}$$

we obtain an explicit expression

$$\frac{\mathrm{d}q}{\mathrm{dt}} = \left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)\cos\epsilon - \left(\frac{\mathrm{d}\overline{\varphi}}{\mathrm{dt}}\right)\sin\epsilon\sin\psi - \left(\frac{\mathrm{d}\overline{\gamma}}{\mathrm{dt}}\right)\left(\sin\epsilon\cos\psi\sin\overline{\varphi} + \cos\epsilon\cos\overline{\varphi}\right), \tag{5}$$

where we used the explicit expressions of the equatorial pole coordinates.

In order to explore an approximate solution of this differential equation, let us ignore the nutation in obliquity when compared with the mean obliquity. Also we assume that the precession in longitude is so small that its cubic and higher order terms are negligible. Further we neglect the difference between two mean obliquities, $\overline{\epsilon}$ and $\overline{\varphi}$, which is small. Then we approximate the right hand side of the above differential equation as

$$\frac{\mathrm{d}q}{\mathrm{dt}} \approx \left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)\cos\overline{\epsilon} - \left(\frac{\mathrm{d}\overline{\epsilon}}{\mathrm{dt}}\right)\psi\sin\overline{\epsilon} - \left(\frac{\mathrm{d}\overline{\gamma}}{\mathrm{dt}}\right) = \frac{\mathrm{d}}{\mathrm{dt}}\left(\psi\cos\overline{\epsilon} - \overline{\gamma}\right). \tag{6}$$

This is easily integrated and, apart from the constant offset, we obtain an approximate solution $q \approx q^*$ where $q^* \equiv \psi\cos\overline{\epsilon} - \overline{\gamma}$. This is further decomposed into the sum of the secular and the periodic components as $q^* = \overline{q^*} + \Delta q^*$, where $\overline{q^*} = \overline{\psi}\cos\overline{\epsilon} - \overline{\gamma}$ and $\Delta q^* = \Delta\psi\cos\overline{\epsilon}$. The former is no other than the accumulated precession in RA and the latter is the well-known nutation in RA. Therefore, the angle q^* can be said as the accumulated precession and nutation in RA. Note that q^* contains no large mixed secular terms of high orders.

Now we change the variable to be integrated from q itself to the deviation from this approximate solution as $\delta q \equiv q - q^*$. Then its differential equation becomes

$$\frac{\mathrm{d}\delta q}{\mathrm{dt}} = \left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)(\cos\epsilon - \cos\overline{\epsilon}) + \left(\frac{\mathrm{d}(\overline{\epsilon} - \overline{\varphi})}{\mathrm{dt}}\right)\psi\sin\overline{\epsilon} + \left(\frac{\mathrm{d}\overline{\varphi}}{\mathrm{dt}}\right)(\psi - \sin\psi)\sin\overline{\epsilon}$$

$$-\left(\frac{\mathrm{d}\overline{\varphi}}{\mathrm{dt}}\right)\sin\psi\,(\sin\epsilon - \sin\overline{\epsilon}) + 2\left(\frac{\mathrm{d}\overline{\gamma}}{\mathrm{dt}}\right)\left[\sin^2\left(\frac{\epsilon - \overline{\varphi}}{2}\right) + \sin^2\frac{\psi}{2}\sin\epsilon\sin\overline{\varphi}\right]. \tag{7}$$

All the terms in the right hand side are small in the sense they are of the order of nutation or of the quadratic and higher terms of precessions.

We numerically integrated the transformed equation. The resulting expression of δq seems to contain no significant nonlinear trend nor mixed secular terms judging from its numerical plotting. By using the successive method of harmonic analysis again, we decomposed the integrated δq into a sum of a quintic polynomial of time and some harmonic terms as $\delta q = \overline{\delta q} + \Delta(\delta q)$. Here the secular part is

$$\overline{\delta q} = -3.857532t - 0.108393t^2 - 0.148087t^3 + 0.014484t^4 - 0.000481t^5, \tag{8}$$

where the unit is mas, t is the time since J2000.0 measured in Julian centuries, and we set the constant term of the secular part be zero for simplicity.

As for the residuals of the decomposition, the rms and the maximum deviation is almost the same as in the case of $s + XY/2$; 0.19 μas and 0.96 μas, respectively. This time, however, this level of precision is realized by the fewer terms. Namely q or δq is more suitable than s or $s + XY/2$ in describing the inertial motion of the location of the CEO. For example, 5 largest terms are enough to approximate δq with the maximum error less than 0.1 mas during the period 1900-2100. While we need 8 terms in expressing $s + XY/2$ at the same level.

Let us return to q itself. We separate it into the sum of the secular and periodic terms; $q = \overline{q} + \Delta q$. Rearranging the approximate formulae of q obtained in the above, we have the final expressions $\overline{q} = \overline{q^*} + \overline{\delta q}$ and $\Delta q = \Delta q^* + \Delta(\delta q)$. In order to obtain the secular part, we evaluated $\overline{q^*}$ for the period 1900-2100 and determined its best-fit polynomial as

$$\overline{q^*} = \overline{\psi}\cos\overline{\epsilon} - \overline{\gamma} = 12.911569 + 4612160.517397t + 1391.650906t^2$$
$$+0.023238t^3 - 0.019475t^4 + 0.000002t^5, \tag{9}$$

where the unit is mas. Combining this and Eq.(8), we separate the secular part $\overline{q}$ into the linear and nonlinear parts as $\overline{q} = \overline{q}_{\mathrm{L}} + \overline{q}_{\mathrm{NL}}$, where

$$\overline{q}_{\mathrm{L}} = 12.911569 + 4612156.659865t, \tag{10}$$

and

$$\overline{q}_{\mathrm{NL}} = 1391.542507t^2 - 0.124849t^3 - 0.004991t^4 - 0.000479t^5. \tag{11}$$

3. RELATIONS AMONG UT1, GMST, AND ERA

The rotation matrix $\mathcal{SNP}$ is written in the post-2003 IAU formulation as $(\mathcal{SNP})_{\mathrm{NEW}} = \mathcal{R}_3(\mathrm{ERA} - s)\mathcal{Q}$. Since the rotation matrix converting the x-axis of the ICRF to the CEO must be the same both in the pre- and the post-2003 IAU formulations, we obtain the identity

$\mathcal{R}_3(-s)\mathcal{Q} = \mathcal{R}_3(q)(\mathcal{NP})_{\rm NEW}$. This means $\mathcal{S}_{\rm NEW} = \mathcal{R}_3({\rm ERA} + q)$. Namely ${\rm GAST_{NEW}} = {\rm ERA} + q$. This relation is separated into those of the secular and periodic parts, respectively, as

$$\mathrm{GMST_{NEW}} = \mathrm{ERA} + \overline{q}, \qquad \mathrm{EE_{NEW}} = \Delta q. \tag{12}$$

Since the ERA is a linear function of UT1, we split the secular part further as

$$(\mathrm{GMST_{NEW}})_\mathrm{L} = \mathrm{ERA} + \overline{q}_\mathrm{L}, \qquad (\mathrm{GMST_{NEW}})_\mathrm{NL} = \overline{q}_\mathrm{NL}. \tag{13}$$

where the subsript L means the linear part and NL does the nonlinear one. We have already obtained the nonlinear part and the periodic part in the previous section.

Now that the relation between the ERA and GMST become clear, there are two ways to fix their relations with the UT1; determining the ERA-UT1 relation first or the GMST-UT1 relation first. Here we take the latter aproach.

Since the nonlinear part is already obtained, we determine the linear part of the GMST-UT1 relation. As was recommended in the IAU 2000 Resolution B 1.8 adopted at the Manchester General Assembly, the continuation of the UT1 in its value and rate must be kept at the transition. Since the polar motion part is unchanged, the condition of continuation is written in the matrix form as

$$(\mathcal{SNP})_\mathrm{IERS1997} = (\mathcal{SNP})_\mathrm{NEW} \tag{14}$$

where the relation must hold in the value and its rate at a certain time. Here the matrix with the subscript IERS1997 is realized by the International Earth Rotation Service (IERS) since 1997.

$$(\mathcal{SNP})_\mathrm{IERS1997} = \mathcal{S}_\mathrm{OBS}\mathcal{N}_\mathrm{OBS}\mathcal{P}_\mathrm{A}, \tag{15}$$

This is different from the genuine form in the pre-2003 IAU formulation

$$(\mathcal{SNP})_\mathrm{IAU1976} = \mathcal{S}_\mathrm{A}\mathcal{N}_\mathrm{A}\mathcal{P}_\mathrm{A}, \tag{16}$$

in the sense that the observed corrections in the nutation, $\delta\psi$ and $\delta\epsilon$, are added in the sidereal rotation matrix and the nutation matrix as

$$\mathcal{S}_\mathrm{OBS} = \mathcal{R}_3\left(\mathrm{GAST_A} + \delta\psi\cos\epsilon_\mathrm{A}\right), \tag{17}$$

and

$$\mathcal{N}_\mathrm{OBS} = \mathcal{R}_1\left(-\epsilon_\mathrm{A} - \Delta_\mathrm{IAU}\epsilon - \delta\epsilon\right)\mathcal{R}_3\left(-\Delta_\mathrm{IAU}\psi - \delta\psi\right)\mathcal{R}_1\left(\epsilon_\mathrm{A}\right), \tag{18}$$

where $\mathcal{P}_\mathrm{A}$ is the precession matrix of Lieske et al. (1977) and $\Delta_\mathrm{IAU}\psi$ and $\Delta_\mathrm{IAU}\epsilon$ are the IAU 1980 nutation.

This convention means that the correction of the precession rate, Δp, of around 3.0 mas/yr, which has been realized by a linear drift in the observed corrections $\delta\psi$, has been already taken into account in determining the UT1 by the IERS 1997 procedure even in the pre-2003 IAU formulation. Therefore, when the precession formulae are replaced by the latest ones including the precession correction, the resulting new GMST-UT1 relation *must* contain the direct contribution of the precession correction. In other words, the numerical coefficients in the linear part of the GMST-UT1 relation would change roughly the same as $\Delta p\cos\epsilon_0 \approx 2.7$ mas/yr when the new precession formulae are applied.

In the previous work (Fukushima 2003), we determined the equinox correction E so as to satisfy the condition

$$\mathcal{N}_\mathrm{OBS}\mathcal{P}_\mathrm{A} = \mathcal{R}_3(E)\,(\mathcal{NP})_\mathrm{NEW}\,. \tag{19}$$

The amount of the equinox correction is as small as

$$E = (+52.403 + 1.452\,t)\ \mathrm{mas} = (+0.00349353 + 0.00009680\,t)\ \mathrm{second}, \tag{20}$$

where t is the time since J2000.0 measured in Julian centuries and we reserved superfluous number of digits after the unit conversion for a later use. See Eq.(39) and Table 2 of Fukushima (2003). Then the above condition of continuation is translated as

$$\mathrm{GAST_A} + (\delta\psi)\cos\epsilon_{\mathrm{A}} = \mathrm{GAST_{NEW}} + E. \tag{21}$$

Since $\delta\psi$ is observbationally determined, this relation must be regarded as a relation to be held in average. In other words, the unknown linear term of $\mathrm{GMST_{NEW}}$ is determined from the observed $\delta\psi$ by a method of weighted least squares. To do this, let us define the quantity

$$\Delta\mathrm{EE} \equiv \mathrm{EE_A} + (\delta\psi)\cos\epsilon_{\mathrm{A}} - \mathrm{EE_{NEW}}. \tag{22}$$

By conducting a method of weighted linear least squares, we determined the best linear function fitted to the data as

$$(\Delta\mathrm{EE})_{\mathrm{L}} = -(11.822 \pm 0.011) - (276.542 \pm 0.187)\,\tau \tag{23}$$

where the unit is mas and τ is the time measured since J1990.0 in Julian century. This is rewritten in terms of t measured since J2000.0 in Julian century as

$$(\Delta\mathrm{EE})_{\mathrm{L}} = (-38.476 - 276.542\,t)\ \mathrm{mas} = (-0.0025651 - 0.0184361\,t)\ \mathrm{second} \tag{24}$$

where we dropped the uncertainties. The weighted rms of the obtained residuals is 0.55 mas.

Then the equation of continuation is rewritten in terms of the secular part as

$$\mathrm{GMST_A}\left(0^{\mathrm{h}}\mathrm{UT1}\right) + (\Delta EE)_{\mathrm{L}} = \mathrm{GMST_{NEW}}\left(0^{\mathrm{h}}\mathrm{UT1}\right) + E. \tag{25}$$

Let us compare the nonlinear part of the both sides. Since the estimated difference ΔEE and the equinox correction E have no nonlinear part, the difference in the nonlinear part is only that in $\overline{q}_{\mathrm{NL}}$. It is as small as

$$(\overline{q}_{\mathrm{NL}})_{\mathrm{A}} - (\overline{q}_{\mathrm{NL}})_{\mathrm{NEW}} = \left(5.017483\,t^2 + 0.031791\,t^3 + 0.004999\,t^4 + 0.000543\,t^5\right)\ \mathrm{mas}$$
$$= \left(0.00033449887\,T_{\mathrm{U}}^2 + 0.0000021194\,T_{\mathrm{U}}^3 + 0.00000033327\,T_{\mathrm{U}}^4 + 0.0000000362\,T_{\mathrm{U}}^5\right)\ \mathrm{second}. \tag{26}$$

After the unit conversion, we again kept more digits than being meningful for the later use. Let us write the linear part of the new GMST as

$$(\mathrm{GMST_{NEW}})_{\mathrm{L}}\left(0^{\mathrm{h}}\mathrm{UT1}\right) = A + BT_{\mathrm{U}}, \tag{27}$$

where A and B are the constants to be determined. Then, the equation to be fitted is rewritten as

$$\begin{aligned} A + B\,T_{\mathrm{U}} &= \mathrm{GMST_A}\left(0^{\mathrm{h}}\mathrm{UT1}\right) + (\Delta\mathrm{EE})_{\mathrm{L}} - E + [(\overline{q}_{\mathrm{NL}})_{\mathrm{A}} - (\overline{q}_{\mathrm{NL}})_{\mathrm{NEW}}] \\ &= 24110.5423514 + 8640184.7943331\,T_{\mathrm{U}} + 0.00033449887\,T_{\mathrm{U}}^2 \\ &\quad +0.0000021194\,T_{\mathrm{U}}^3 + 0.00000033327\,T_{\mathrm{U}}^4 + 0.0000000362\,T_{\mathrm{U}}^5. \end{aligned} \tag{28}$$

If we write the right hand side formally as $\sum_k a_k T_{\mathrm{U}}^k$, then the condition of continuation is solved as

$$A = \sum_k (1-k) a_k T_{\mathrm{F}}^k, \qquad B = \sum_k k a_k T_{\mathrm{F}}^{k-1} \tag{29}$$

where T_{F} is the time of fitting. Namely

$$A = 24110.5423514 - 0.00033450\,T_{\mathrm{F}}^2 - 0.00000424\,T_{\mathrm{F}}^3 - 0.00000100\,T_{\mathrm{F}}^4 - 0.00000014\,T_{\mathrm{F}}^5, \tag{30}$$

$$B = 8640184.7943331 + 0.00066900\,T_{\rm F} + 0.00000636\,T_{\rm F}^2 + 0.00000133\,T_{\rm F}^3 + 0.00000018\,T_{\rm F}^4. \quad (31)$$

Since the ERA is connected to the linear part of GMST, we obtain the relation between ERA and UT1 as a direct byproduct as $\mathrm{ERA}_{\rm NEW} = A + BT_{\rm U} - \overline{q}_{\rm L}$. As another byproduct, we obtain the ratio of rates of the universal and sidereal times as

$$r_{\rm NEW} \equiv 1 + \frac{1}{86400 \times 36525} \left(\frac{\mathrm{dGMST}_{\rm NEW}\,(0^{\rm h}\mathrm{UT1})}{\mathrm{d}T_{\rm U}} \right)$$

$$= r_0 + 2.11993 \times 10^{-13}\,T_{\rm U} + 2.015 \times 10^{-15}\,T_{\rm U}^2 + 4.21 \times 10^{-16}\,T_{\rm U}^3 + 5.7 \times 10^{-17}\,T_{\rm U}^4, \qquad (32)$$

where $r_0 \equiv 1 + B/3155760000$.

4. CONCLUSION

By using our precession theory (Fukushima 2003) and nutation series (Shirai and Fukushima 2001), we confirmed that the accumulated precession and nutation in RA, q^*, is a good approximation of q. Then we transformed the differential equation of q into that of $\delta q \equiv q - q^*$. By integrating the transformed differential equation numerically, we obtained δq for the period 1900-2100. Third, by using a successive method of harmonic analysis, we decomposed the quantity δq into a low order polynomial and some periodic and mixed secular terms. By comparing the manner of decrease of the approximation errors with respect to the number of terms, we judge that δq is more suitable than $s + XY/2$ to specify the location of the CEO on the true equator of date.

Then, by using thus-determined formulae of q^* and δq, we determined the new formulas relating Greenwich Apparent Sidereal Timte (GAST), Earth Rotation Angle (ERA), Greenwich Mean Sidereal Time (GMST), Equation of Equinox (EE), and UT1 from the continuation condition that the values and rates of UT1 determined by the pre-2003 IERS procedure and by the new procedure are the same at a certain time of fitting. The resulting formulas are symbolically expressed as

$$\mathrm{GAST}_{\rm NEW} = \mathrm{GMST}_{\rm NEW} + \mathrm{EE}_{\rm NEW}, \qquad \mathrm{ERA}_{\rm NEW} = A + BT_{\rm U} - \overline{q}_{\rm L},$$

$$\mathrm{GMST}_{\rm NEW} = A + BT_{\rm U} + \overline{q}_{\rm NL}, \qquad \mathrm{EE}_{\rm NEW} = (\Delta\psi)\cos\overline{\epsilon} + \Delta(\delta q) \qquad (33)$$

where $\overline{q}_{\rm L}$ and $\overline{q}_{\rm NL}$ are the linear and the nonlinear terms of the secular part of q given in the above, respectively, and $\Delta(\delta q)$ is the periodic part of δq.

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