TIME TRANSFER AND FREQUENCY SHIFT UP TO THE ORDER $1/c^4$ IN THE FIELD OF AN AXISYMMETRIC ROTATING BODY

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ABSTRACT. We present a general procedure to determine up to the order $1/c^4$ the relativistic influence of the mass and spin multipoles on the time transfers and the frequency shifts in the vicinity of an isolated, axisymmetric rotating body. This procedure is applied within the Nordtvedt-Will parametrized post-Newtonian formalism. We give explicit formulae for the contributions of the mass, of the quadrupole moment and of the angular momentum of the rotating body.

1. INTRODUCTION

The art of ultraprecise timekeeping is rapidly progressing. The laser-cooled atomic clock PHARAO scheduled to fly on the International Space Station (ISS) in 2006 or 2007 is expected to stick with an accuracy of 10^{-16} (ESA/ACES mission). The NIST/NASA/JPL PARCS experiment scheduled for launch in 2007 is projected to enter the 5×10^{-17} accuracy range. And new kinds of optical clocks extracting time from calcium atoms or mercury ions are expected to reach an accuracy of the order of 10^{-18} in the foreseeable future.

At a level of uncertainty about 10^{-18}, a fully relativistic calculation of the time and frequency transfers must be performed up to the order $1/c^4$ within the post-Newtonian formalism. We present here the results that we have recently obtained at this order of approximation for an isolated, axisymmetric rotating body having a stationary gravitational field (Linet and Teyssandier 2002, denoted by LT (2002) in what follows). The problem is treated within the Nordtvedt-Will parametrized post-Newtonian (PPN) formalism (Will 1993). It seems that in the previous works devoted to the time and frequency transfers between a satellite and the ground, the calculations were carried out only up to the order $1/c^3$, in the narrow context of general relativity (see Blanchet et al. (2001) and Refs. therein).

Our procedure gives means of determining the contributions of all the mass and spin multipoles of the rotating body. We obtain explicit expressions for the contributions of the mass, of the quadrupole moment and of the angular momentum. The numerical estimates are performed for a photon emitted from a satellite A orbiting at the altitude $h = 400$ km and received by a terrestrial station B.

2. THE WORLD FUNCTION

We use the method of the world function (Synge 1964), which presents the great advantage to spare the trouble of solving the differential equations of light rays. So we recall the definition and the fundamental properties of this function. Spacetime is assumed to be covered by a global quasi-Cartesian coordinate system $(x^\mu) = (x^0, \boldsymbol{x})$, with $x^0 = ct$. The metric is denoted by $g_{\mu\nu}$. The signature of the metric is -2.

Consider two points $x_A = (ct_A, \boldsymbol{x}_A)$ and $x_B = (ct_B, \boldsymbol{x}_B)$. If these points are close enough to one another, they are connected by a unique geodesic path Γ_{AB}. The world function is the two-point function $\Omega(x_A, x_B)$ defined by

$$\Omega(x_A, x_B) = \frac{1}{2}\varepsilon_{AB}\left[s_{AB}\right]^2 , \tag{1}$$

where s_{AB} is the geodesic distance between x_A and x_B and $\varepsilon_{AB} = 1, 0, -1$ according to whether Γ_{AB} is a timelike, null or spacelike geodesic, respectively.

The world function has the following properties.

i) In a vacuum, a light ray is a null geodesic of the metric g. As a consequence, two points x_A and x_B are linked by a light ray if and only if the condition

$$\Omega(x_A, x_B) \equiv \Omega(ct_A, \boldsymbol{x}_A, ct_B, \boldsymbol{x}_B) = 0 \tag{2}$$

is fulfilled. Solving this equation for t_B yields the travel time $t_B - t_A$ of a photon emitted at x_A and received at x_B as a function of t_A, $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$: $t_B - t_A = \mathcal{T}(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B)$. This function is called the time transfer function.

ii) Let x_A and x_B be two points connected by a null geodesic Γ_{AB}. The covariant components of the vector $p^\alpha = dx^\alpha/d\lambda$ tangent to Γ_{AB}, respectively, at x_A and x_B are given by

$$(p_\alpha)_A \equiv \left(g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right)_A = -\frac{\partial\Omega}{\partial x_A^\alpha}\,, \qquad (p_\alpha)_B \equiv \left(g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right)_B = \frac{\partial\Omega}{\partial x_B^\alpha}\,, \tag{3}$$

where λ is the unique affine parameter along Γ_{AB} such that $\lambda_A = 0$ and $\lambda_B = 1$.

Thus, if $\Omega(x_A, x_B)$ is known, it is possible to determine the frequency shift between x_A and x_B. Indeed, let ν_A be the proper frequency of a photon as measured by an observer A at x_A moving with the unit 4-velocity $u_A^\alpha = (dx^\alpha/ds)_A$ and ν_B be the proper frequency of the same photon as measured by an observer B at x_B moving with the unit 4-velocity $u_B^\alpha = (dx^\alpha/ds)_B$. The frequency shift $\nu_A/\nu_B - 1$ is given by the well-known formula

$$\frac{\nu_A}{\nu_B} - 1 = \frac{u_A^\alpha (p_\alpha)_A}{u_B^\alpha (p_\alpha)_A} - 1\,. \tag{4}$$

In what follows, we suppose that the light rays are propagating in a stationary gravitational field generated by an isolated, axisymmetric rotating body. The coordinates are chosen so that the metric does not depend on x^0. Then the world function is of the form $\Omega(x_B^0 - x_A^0, \boldsymbol{x}_A, \boldsymbol{x}_B)$ and the travel time of a photon propagating from x_A to x_B may be written as

$$t_B - t_A = \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)\,. \tag{5}$$

If the time transfer function $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ is known, it is easy to determine the frequency shift between A and B since it can be shown that Eq. (4) reduces to

$$\frac{\nu_A}{\nu_B} - 1 = \frac{u_A^0}{u_B^0} \times \frac{1 + \boldsymbol{v}_A \cdot \nabla_{\boldsymbol{x}_A}\mathcal{T}}{1 - \boldsymbol{v}_B \cdot \nabla_{\boldsymbol{x}_B}\mathcal{T}} - 1\,, \tag{6}$$

where $\boldsymbol{v}_A = (d\boldsymbol{x}/dt)_A$ and $\boldsymbol{v}_B = (d\boldsymbol{x}/dt)_B$ are the coordinate velocities of the clocks comoving with A and B, respectively.

3. TIME TRANSFER AND FREQUENCY SHIFT

The center of mass O of the rotating body is taken as the origin of the coordinates $(\boldsymbol{x})$, Ox^3 being both the axis of symmetry and the axis of rotation. The angular momentum of the body about Ox^3 is denoted by $\boldsymbol{S}$. The unit vector along the x^3-axis is denoted by $\boldsymbol{k}$. We write $\boldsymbol{S} = S\boldsymbol{k}$. We put $r = |\boldsymbol{x}|$, $r_A = |\boldsymbol{x}_A|$, $r_B = |\boldsymbol{x}_B|$. We denote by r_e the equatorial radius. We call θ the angle between $\boldsymbol{x}$ and $\boldsymbol{k}$. We denote by $\boldsymbol{v}_r$ the velocity of the center of mass O relative to the universe rest frame.

With a convenient choice of coordinates, the PPN metric may be written in the form

$$g_{00} = 1 - \frac{2}{c^2}W + \frac{2\beta}{c^4}W^2 + \frac{1}{c^4}f(\xi, \alpha_2, \alpha_3, \zeta_1, ..., \zeta_4) + O(6)\,, \tag{7}$$

$$\{g_{0i}\} = \frac{2}{c^3}\left[(\gamma + 1 + \frac{1}{4}\alpha_1)\boldsymbol{W} + \frac{1}{4}\alpha_1 \boldsymbol{v}_r\, W\right] + O(5)\,, \tag{8}$$

$$g_{ij} = -\left(1 + \frac{2\gamma}{c^2}\,W\right)\delta_{ij} + O(4)\,, \tag{9}$$

where $f(\xi, \alpha_2, \alpha_3, \zeta_1, ..., \zeta_4)$ denotes contributions involving post-Newtonian parameters that we do not take into account here and W and $\boldsymbol{W}$ are potentials which may be expanded in multipole series of the form

$$W(\boldsymbol{x}) = \frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{r_e}{r}\right)^n P_n(\cos\theta)\right]\,, \tag{10}$$

$$\boldsymbol{W}(\boldsymbol{x}) = \frac{G\boldsymbol{S}\times\boldsymbol{x}}{2r^3}\left[1 - \sum_{n=1}^{\infty} K_n \left(\frac{r_e}{r}\right)^n P'_{n+1}(\cos\theta)\right] \tag{11}$$

in the region $r > r_e$.

In these equations, the P_n are the Legendre polynomials; M, J_2, ..., J_n, ..., $\boldsymbol{S}$, ..., K_n, ... correspond to the generalized Blanchet-Damour mass and spin multipole moments (Blanchet 1989, Damour et al 1991 , Klioner and Soffel 2000). Note that the coefficients K_n in Eq. (11) coincide up to $1/c^2$ terms with the spin multipole moments calculated by one of us (Teyssandier 1977 and 1978).

Putting $\boldsymbol{R}_{AB} = \boldsymbol{x}_B - \boldsymbol{x}_A$ and $R_{AB} = |\,\boldsymbol{x}_B - \boldsymbol{x}_A\,|$, we have shown that the world function $\Omega(x_A, x_B)$ may be written in the form

$$\begin{aligned}\Omega(x_A, x_B) = {} & \frac{1}{2}\left[(x^0_B - x^0_A)^2 - R^2_{AB}\right] - \frac{1}{c^2}\left[(x^0_B - x^0_A)^2 + \gamma R^2_{AB}\right]\mathcal{W}(\boldsymbol{x}_A, \boldsymbol{x}_B) \\ & + \frac{2}{c^3}(\gamma + 1 + \frac{1}{4}\alpha_1)(x^0_B - x^0_A)\boldsymbol{R}_{AB}\cdot \boldsymbol{\mathcal{W}}(\boldsymbol{x}_A, \boldsymbol{x}_B) \\ & + \frac{1}{2c^3}\alpha_1(x^0_B - x^0_A)(\boldsymbol{R}_{AB}\cdot\boldsymbol{v}_r)\mathcal{W}(\boldsymbol{x}_A, \boldsymbol{x}_B) + O(4)\,,\end{aligned} \tag{12}$$

where $\mathcal{W}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ and $\boldsymbol{\mathcal{W}}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ are given by the following multipole expansions when $r > r_e$:

$$\mathcal{W}(\boldsymbol{x}_A, \boldsymbol{x}_B) = GM\left[1 - \sum_{n=2}^{\infty}\frac{1}{n!}J_n r_e^n \frac{\partial^n}{\partial z^n}\right] F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B)\Big|_{\boldsymbol{x}=0}\,, \tag{13}$$

$$\boldsymbol{\mathcal{W}}(\boldsymbol{x}_A, \boldsymbol{x}_B) = -\frac{1}{2}G\boldsymbol{S}\times\nabla\left[1 - \sum_{n=1}^{\infty}\frac{1}{n!}K_n r_e^n \frac{\partial^n}{\partial z^n}\right] F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B)\Big|_{\boldsymbol{x}=0}\,, \tag{14}$$

the kernel function $F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B)$ being defined as

$$F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{R_{AB}} \ln \left(\frac{\mid \boldsymbol{x} - \boldsymbol{x}_A \mid + \mid \boldsymbol{x} - \boldsymbol{x}_B \mid + R_{AB}}{\mid \boldsymbol{x} - \boldsymbol{x}_A \mid + \mid \boldsymbol{x} - \boldsymbol{x}_B \mid - R_{AB}} \right) . \tag{15}$$

These formulae show that the multipole expansion of $\Omega(x_A, x_B)$ can be thoroughly calculated up to the order $1/c^3$ by straightforward differentiations of the kernel function given by (15). Note that integral expressions of $\mathcal{W}$ and $\boldsymbol{\mathcal{W}}$ valid everywhere are also given in LT (2002).

We have shown that the time transfer $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ and its multipole expansion can be explicitly calculated up to the order $1/c^4$ when $\Omega(x_A, x_B)$ is known up to the order $1/c^3$. In the present communication, we retain only the contributions due to M, J_2 and $\boldsymbol{S}$. Thus, putting

$$\boldsymbol{n}_A = \frac{\boldsymbol{x}_A}{r_A}, \quad \boldsymbol{n}_B = \frac{\boldsymbol{x}_B}{r_B}, \quad \boldsymbol{N}_{AB} = \frac{\boldsymbol{x}_B - \boldsymbol{x}_A}{R_{AB}}, \tag{16}$$

and using the identity $(r_A + r_B)^2 - R_{AB}^2 = 2r_A r_B(1 + \boldsymbol{n}_A \cdot \boldsymbol{n}_B)$, we get

$$\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{c} R_{AB} + \mathcal{T}_M(\boldsymbol{x}_A, \boldsymbol{x}_B) + \mathcal{T}_{J_2}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \mathcal{T}_{\boldsymbol{S}}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \mathcal{T}_{\boldsymbol{v}_r}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \cdots ,$$

where

$$\mathcal{T}_M(\boldsymbol{x}_A, \boldsymbol{x}_B) = (\gamma + 1) \frac{GM}{c^3} \ln \left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right) , \tag{17}$$

$$\mathcal{T}_{J_2}(\boldsymbol{x}_A, \boldsymbol{x}_B) = -\frac{\gamma + 1}{2} \frac{GM}{c^3} \frac{J_2 r_e^2}{r_A r_B} \frac{R_{AB}}{1 + \boldsymbol{n}_A \cdot \boldsymbol{n}_B} \left[\left(\frac{1}{r_A} + \frac{1}{r_B} \right) \frac{(\boldsymbol{k} \cdot \boldsymbol{n}_A + \boldsymbol{k} \cdot \boldsymbol{n}_B)^2}{1 + \boldsymbol{n}_A \cdot \boldsymbol{n}_B} - \frac{(\boldsymbol{k} \times \boldsymbol{n}_A)^2}{r_A} - \frac{(\boldsymbol{k} \times \boldsymbol{n}_B)^2}{r_B} \right] , \tag{18}$$

$$\mathcal{T}_{\boldsymbol{S}}(\boldsymbol{x}_A, \boldsymbol{x}_B) = -(\gamma + 1 + \frac{1}{4}\alpha_1) \frac{GS}{c^4} \left(\frac{1}{r_A} + \frac{1}{r_B} \right) \frac{\boldsymbol{k} \cdot (\boldsymbol{n}_A \times \boldsymbol{n}_B)}{1 + \boldsymbol{n}_A \cdot \boldsymbol{n}_B} , \tag{19}$$

$$\mathcal{T}_{\boldsymbol{v}_r}(\boldsymbol{x}_A, \boldsymbol{x}_B) = -\alpha_1 \frac{GM}{2c^4} (\boldsymbol{N}_{AB} \cdot \boldsymbol{v}_r) \ln \left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right) . \tag{20}$$

The term of order $1/c^3$ given by (17) is the well-known Shapiro time delay. Equations (18) and (19) extend results previously found for $\gamma = 1$ and $\alpha_1 = 0$ (Klioner 1991, Klioner and Kopeikin 1992 ; see also, e.g., Ciufolini et al 2003 and Refs. therein).

Using Eq. (6) and Eqs. (17)-(20), it is possible to perform the calculation of the ratio ν_A/ν_B up to the order $1/c^4$ if the terms of the same order in g_{00} are taken into account. For the sake of simplicity, we henceforth confine ourselves to the fully conservative metric theories of gravity without preferred location effects, in which all the PPN parameters vanish except β and γ. Since the gravitational field is assumed to be stationary, the chosen coordinate system is then a standard post-Newtonian gauge and the metric takes its usual post-Newtonian form (Klioner and Soffel 2000). Then, retaining only the contributions due to M, J_2 and $\boldsymbol{S}$ in the terms of orders $1/c^3$ and $1/c^4$, we have found that the frequency shift $\nu_A/\nu_B - 1$ may be written as

$$\frac{\nu_A}{\nu_B} - 1 = \left(\frac{\delta\nu}{\nu} \right)_c + \left(\frac{\delta\nu}{\nu} \right)_g , \tag{21}$$

where $(\delta\nu/\nu)_c$ is the special relativistic Doppler effect given by

$$\left(\frac{\delta\nu}{\nu}\right)_c = -\frac{1}{c}\left[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)\right]+\frac{1}{c^2}\left[\frac{1}{2}v_A^2-\frac{1}{2}v_B^2-[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)](\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\right]$$
$$-\frac{1}{c^3}[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\left[\frac{1}{2}v_A^2-\frac{1}{2}v_B^2+(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)^2\right] \tag{22}$$
$$+\frac{1}{c^4}\left\{\frac{3}{8}v_A^4-\frac{1}{4}v_A^2v_B^2-\frac{1}{8}v_B^4\right.$$
$$\left.-[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)](\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\left[\frac{1}{2}v_A^2-\frac{1}{2}v_B^2+(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)^2\right]\right\}+O(5)$$

and $(\delta\nu/\nu)_g$ is the gravitational part given by

$$\left(\frac{\delta\nu}{\nu}\right)_g=\frac{1}{c^2}(W_A-W_B)+\frac{1}{c^3}\left(\frac{\delta\nu}{\nu}\right)_M^{(3)}+\frac{1}{c^3}\left(\frac{\delta\nu}{\nu}\right)_{J_2}^{(3)}+\frac{1}{c^4}\left(\frac{\delta\nu}{\nu}\right)_M^{(4)}+\frac{1}{c^4}\left(\frac{\delta\nu}{\nu}\right)_{\boldsymbol{S}}^{(4)}+\cdots,$$

the different terms being separately made explicit and briefly estimated in what follows.

The term $(\delta\nu/\nu)_M^{(3)}$ due to the mass reads

$$\left(\frac{\delta\nu}{\nu}\right)_M^{(3)} = -GM\left(\frac{1}{r_A}+\frac{1}{r_B}\right)\left\{\left(\frac{\gamma+1}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{r_A-r_B}{r_A+r_B}\right)[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\right.$$
$$\left.+(\gamma+1)\frac{R_{AB}}{r_A+r_B}\frac{\boldsymbol{n}_A\cdot\boldsymbol{v}_A+\boldsymbol{n}_B\cdot\boldsymbol{v}_B}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\right\}. \tag{23}$$

With K_{AB} defined as

$$K_{AB}=\frac{(r_A-r_B)^2}{r_Ar_B} \tag{24}$$

the term $(\delta\nu/\nu)^{(3)}_{J_2}$ may be written as

$$\begin{aligned}
\left(\frac{\delta\nu}{\nu}\right)^{(3)}_{J_2} &= \frac{GMJ_2}{2r_e}[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\left[\left(\frac{r_e}{r_A}\right)^3\left[3(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2-1\right]-\left(\frac{r_e}{r_B}\right)^3\left[3(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2-1\right]\right] \\
&+\frac{\gamma+1}{2}\frac{GM\,J_2r_e^2(r_A+r_B)}{r_A^2r_B^2}\frac{1}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2} \\
&\times\Bigg\{[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\Bigg[(\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2\frac{5-3\boldsymbol{n}_A\cdot\boldsymbol{n}_B+2K_{AB}}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B} \\
&-\left(1-\frac{r_A(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2+r_B(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A+r_B}\right)(3-\boldsymbol{n}_A\cdot\boldsymbol{n}_B+K_{AB})\Bigg] \\
&+\frac{R_{AB}}{r_A+r_B}(\boldsymbol{n}_A\cdot\boldsymbol{v}_A+\boldsymbol{n}_B\cdot\boldsymbol{v}_B)(\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2\frac{7-\boldsymbol{n}_A\cdot\boldsymbol{n}_B+2K_{AB}}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B} \\
&-\frac{R_{AB}}{r_A}(\boldsymbol{n}_A\cdot\boldsymbol{v}_A)\left[1-3(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2\right]\frac{r_A+r_B(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)}{r_A+r_B} \\
&-\frac{R_{AB}}{r_B}(\boldsymbol{n}_B\cdot\boldsymbol{v}_B)\left[1-3(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2\right]\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B} \qquad (25) \\
&+R_{AB}\Bigg[2\left(\frac{\boldsymbol{n}_A\cdot\boldsymbol{v}_A}{r_A}+\frac{\boldsymbol{n}_B\cdot\boldsymbol{v}_B}{r_B}\right)(\boldsymbol{k}\cdot\boldsymbol{n}_A)(\boldsymbol{k}\cdot\boldsymbol{n}_B) \\
&-(\boldsymbol{n}_A\cdot\boldsymbol{v}_A)\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{r_B}-(\boldsymbol{n}_B\cdot\boldsymbol{v}_B)\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A}\Bigg] \\
&-2\frac{R_{AB}}{r_A}(\boldsymbol{k}\cdot\boldsymbol{v}_A)\left[\boldsymbol{k}\cdot\boldsymbol{n}_A\frac{r_A+r_B(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)}{r_A+r_B}+\boldsymbol{k}\cdot\boldsymbol{n}_B\right] \\
&-2\frac{R_{AB}}{r_B}(\boldsymbol{k}\cdot\boldsymbol{v}_B)\left[\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B}\right]\Bigg\}.
\end{aligned}$$

A crude estimate for the ISS shows that if $\gamma=1$, then

$$\left|\frac{1}{c^3}\left(\frac{\delta\nu}{\nu}\right)^{(3)}_{J_2}\right| \leq 1.3\times10^{-16}\,. \qquad (26)$$

So, it will perhaps be necessary to take into account this effect of J_2 in ACES or PARCS experiments. However, a lower bound will be found if the inclination of the orbit of the ISS and the latitude of the terrestrial station are taken into account.

At the order $1/c^4$, the contribution of the mass is given by

$$\begin{aligned}
\left(\frac{\delta\nu}{\nu}\right)^{(4)}_{M} &= (\gamma+1)\left(\frac{GM}{r_A}v_A^2-\frac{GM}{r_B}v_B^2\right)-\frac{1}{2}\frac{GM(r_A-r_B)}{r_Ar_B}(v_A^2-v_B^2) \qquad (27) \\
&-GM\left(\frac{1}{r_A}+\frac{1}{r_B}\right)\Bigg[\left(\frac{2(\gamma+1)}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{r_A-r_B}{r_A+r_B}\right)[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)](\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B) \\
&+\frac{\gamma+1}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\frac{R_{AB}}{r_A+r_B}\{(\boldsymbol{n}_A\cdot\boldsymbol{v}_A)(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)-[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-2\boldsymbol{v}_B)](\boldsymbol{n}_B\cdot\boldsymbol{v}_B)\}\Bigg] \\
&+\frac{1}{2}\left(\frac{GM}{r_Ar_B}\right)^2\left[(r_A-r_B)^2+2(\beta-1)(r_A^2-r_B^2)\right]\,.
\end{aligned}$$

The dominant term $(\gamma+1)GM/r_Av_A^2$ in (27) induces a correction to the frequency shift which amounts to 10^{-18}. So, it will certainly be necessary to take this term into account in experiments performed in the foreseeable future.

Finally, the term $c^{-4}(\delta\nu/\nu)^{(4)}_{\boldsymbol{S}}$ due to the angular momentum is

$$\left(\frac{\delta\nu}{\nu}\right)^{(4)}_{\boldsymbol{S}} = (\gamma+1)\,\frac{GS}{r_A r_B}\,\boldsymbol{v}_A\cdot\left\{\left(1+\frac{r_B}{r_A}\right)\frac{\boldsymbol{k}\times\boldsymbol{n}_B}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{r_B}{r_A}\,(\boldsymbol{k}\times\boldsymbol{n}_A)\right.$$
$$\left.+\frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2}\left[\left(1+\frac{r_B}{r_A}(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)\right)\boldsymbol{n}_A+\left(1+\frac{r_B}{r_A}\right)\boldsymbol{n}_B\right]\right\} \quad (28)$$
$$-(\gamma+1)\,\frac{GS}{r_A r_B}\,\boldsymbol{v}_B\cdot\left\{\left(1+\frac{r_A}{r_B}\right)\frac{\boldsymbol{k}\times\boldsymbol{n}_A}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{r_A}{r_B}\,(\boldsymbol{k}\times\boldsymbol{n}_B)\right.$$
$$\left.-\frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2}\left[\left(1+\frac{r_A}{r_B}(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)\right)\boldsymbol{n}_B+\left(1+\frac{r_A}{r_B}\right)\boldsymbol{n}_A\right]\right\}.$$

Taking for the Earth $S = 5.86\times10^{33}$ kg m^2 s^{-1}, we obtain the inequality

$$\left|\frac{1}{c^4}\left(\frac{\delta\nu}{\nu}\right)^{(4)}_{\boldsymbol{S}}\right| \leq (\gamma+1)\times10^{-19}\,. \quad (29)$$

Thus, our formula confirms that the effect of the angular momentum of the Earth on the frequency shift will not affect the ACES or PARCS experiments.

4. REFERENCES

Blanchet, L., Salomon, C., Teyssandier, P. and Wolf, P., 2001, *Astron. Astrophys.* **370**, 320.

Blanchet, L. and Damour, T., 1989, Ann. Inst. H. Poincaré Phys. Théor. **50**, 377.

Ciufolini, I., Kopeikin, S. M., Mashhoon, B., and Ricci, F., Phys. Lett. A, 2003, **308**, 101.

Damour, T., Soffel, M. and Xu, C., 1991, *Phys. Rev. D* **43**, 3273.

Klioner, S. A., 1991, *Sov. Astron.* **35**, 523.

Klioner, S. A. and Kopeikin, S. M., 1992, *Astron. J.* **104**, 897.

Klioner, S. A. and Soffel, M., 2000 *Phys. Rev. D* **62**, 024019.

Linet, B. and Teyssandier, P., 2002, *Phys. Rev. D* **66**, 024045.

Synge, J. L., 1964, *Relativity: the General Theory* (North-Holland).

Teyssandier, P., 1977, *Phys. Rev. D* **16**, 946; 1978, *Phys. Rev. D* **18**, 1037.

Will, C. 1993, *Theory and Experiment in Gravitational Physics*, revised edition (Cambridge University Press).