FREE MOTION OF ELASTIC BODIES WITH RESPECT TO AN INERTIAL AND BODY-FIXED FRAME. APPLICATION TO THE EARTH

J. SOUCHAY[1] and M. FOLGUEIRA[2]

[1] SYRTE - Observatoire de Paris
61 avenue de l'Observatoire
E-mail: Jean.Souchay@obspm.fr

[2] Instituto de Astronomía y Geodesia (UCM-CSIC), Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid, 28040 Madrid. Spain
E-mail: martafl@iagmat1.mat.ucm.es

ABSTRACT

The free motion of an elastic body can be obtained from that of a rigid one with the same parameters, but by taking into account the variations of the components of the tensor of inertia with respect to the time.

We show how this way of computation, constructed either from the classical equations for the angular momentum or from Hamiltonian theory as well, leads to some interesting contributions to the polar motion, depending on the elasticity and the triaxial form of the body (for instance the Earth), to be added to some classical ones, as the presence of the Chandler wobble.

1. INTRODUCTION.

The torque-free rotational motion, sometimes called the *Euler-Poinsot problem*, of a body corresponds to that in which the torque exerted by the Sun and the Moon is neglected. The formulation of the corresponding equations of such problem for elastic body can be expressed into two different ways:

- In terms of the rectangular components of the angular velocity vector, by considering the classical Liouville equations, and
- In terms of Andoyer's variables (L, G, H) and its canonically conjugate variables (l, g, h), by using Hamiltonian mechanics.

Motivated by Kubo's procedure (1991), which applied Hamiltonian mechanics to the problem of the free motion of the elastic Earth, we undertake an extension of his study considering an axially body instead of a body with rotational symmetry.

2. EXTENSION OF KUBO'S PROCEDURE: VARIATIONAL EQUATIONS.

In this section, we will generalize the equations of the torque-free rotational motion of a rigid body in terms of the Andoyer variables as discussed in (Souchay *et al.*, 2002), to the case of a deformable body. The Hamiltonian for the free rotation (which is equal to the kinetic energy) in this case has the form (Kubo, 1991):

$$\begin{aligned} \mathcal{K} &= \frac{1}{2\,ABC}\Big\{\Big[\,BC\sin^2 l \,+\, AC\cos^2 l\Big]\,(G^2-L^2)\,+\,ABL^2 \\ &-\; CF(G^2-L^2)\sin 2l \,-\, 2L\sqrt{G^2-L^2}\,\Big[\,BE\sin l \,+\, AD\cos l\Big]\;\Big\} \end{aligned} \tag{1}$$

where, A, B and C are the moments of inertia and D, E and F are the products of inertia of an non-rigid body with respect to the Tisserand axes ($A = A_0 + \Delta A$, $B = B_0 + \Delta B$ and $C = C_0 + \Delta C$, with $(A_0,\, B_0,\, C_0)$ corresponding to a rigid axially body).

If the elastic body rotates about an axis which deviates from the axis of symmetry of the body, then centrifugal forces tend to distort it and therefore this distorsion originates variations in the tensor of inertia. This effect is known as *rotational deformation*, which is the only one to take into account in the study of the torque-free rotational motion of an elastic body. Periodic variations in the tensor arising from rotational deformation are given by (Kubo, 1991; Souchay & Folgueira, 2002):

$$\begin{aligned} \Delta A &= \Delta B = \Delta C = 0 \\ D &= \beta\, C_0 \sin J^\star\, \cos l^\star \,=\, \beta\, C_0 \sin J^\star\, \cos(l+\delta) \\ E &= \alpha\, C_0 \sin J^\star\, \sin l^\star \,=\, \alpha\, C_0 \sin J^\star\, \sin(l+\delta) \\ F &= 0 \end{aligned} \tag{2}$$

where the following notations were adopted:

$$\alpha \;=\; \frac{k}{k_s}\frac{C_0-A_0}{A_0} \qquad \text{and} \qquad \beta = \frac{k}{k_s}\frac{C_0-A_0}{B_0} \tag{3}$$

with, k is a *Love number*, k_s is the *secular Love number* equals to $\frac{3\kappa^2(C_0-A_0)}{a^5\Omega^2}$ (κ^2 denotes the gravitational constant and a is the Earth's equatorial radius). Following (Kubo, 1991; p.171), D and E should be considered only as functions of time. So, we have denoted J and l in the above expression as $J^\star$ and $l^\star$. Numerically, they are equal to J and and l, respectively. δ represents a time lag between the rotational axis and the pole of the equatorial bulge due to the centrifugal force, then, $l^\star$ should be equal to $l+\delta$ ($\delta > 0$).

When these last expressions are substituted in the general expression of the Hamiltonian (1) one obtains, neglecting terms of second and higher order:

$$\begin{aligned} \mathcal{K} &= \frac{1}{2\,A_0B_0C_0}\left\{\left[B_0C_0\sin^2 l \,+\, A_0C_0\cos^2 l\right]\,(G^2-L^2)\,+\,A_0B_0L^2 \right.\\ &-\; \left. 2L\,\sqrt{G^2-L^2}\,\Big[\,\alpha B_0C_0\sin J^\star\,\sin l^\star\,\sin l \,+\, \beta A_0C_0\sin J^\star\,\cos l^\star\,\cos l\Big]\right\} \end{aligned} \tag{4}$$

Adopting the following notations:

$$\bar{A} = \frac{A_0 + B_0}{2} \qquad \text{and} \qquad \varepsilon = \frac{A_0 - B_0}{A_0 + B_0} \tag{5}$$

the Hamiltonian (4) can be then rewrite as, in terms of $\bar{A}$ and ε:

$$\mathcal{K} = \mathcal{K}^R - \frac{L}{\bar{A}} \sqrt{G^2 - L^2} \sin J^\star \Big[\alpha(1-\varepsilon) \sin l^\star \sin l + \beta(1+\varepsilon) \cos l^\star \cos l \Big] \tag{6}$$

where $\mathcal{K}^R$ is the Hamiltonian of the torque-free motion for a rigid body with a triaxial form (Souchay *et al.*, 2002).

Variational equations:

Once we have obtained the Hamiltonian corresponding to the torque-free rotational motion of an elastic body in terms of Andoyer variables (L, G, H, l, g, h), we can use the general Hamilton's equations of motion to establish the variational equations of the problem considered here (Kinoshita, 1977):

$$\begin{aligned} \frac{d}{dt}(L, G, H) &= -\frac{\partial \mathcal{K}}{\partial(l, g, h)} \\ \frac{d}{dt}(l, g, h) &= \frac{\partial \mathcal{K}}{\partial(L, G, H)} \end{aligned} \tag{7}$$

which express the time variations of Andoyer variables in function of partial derivatives of the Hamiltonian. Thus, the substitution of the Hamiltonian (6) in the previous expressions gives us the following expressions for the temporal variations of Andoyer variables and the angle J:

$$\begin{aligned} \frac{dL}{dt} &= \frac{\varepsilon}{\bar{A}} G^2 \sin^2 J \sin 2l + \frac{GL}{\bar{A}} \sin J \sin J^\star \left[\rho \sin\delta + \tau \sin(2l+\delta)\right] \\ \frac{dG}{dt} &= 0 \\ \frac{dH}{dt} &= 0 \\ \frac{dl}{dt} &= -\frac{L}{\bar{A}} \left(\frac{C_0 - \bar{A}}{C_0} \right) - \frac{\varepsilon}{\bar{A}} L \cos 2l + \frac{L}{\bar{A}} \left[\rho \cos\delta - \tau \cos(2l+\delta)\right] \\ \frac{dg}{dt} &= \frac{G}{\bar{A}} + \frac{\varepsilon}{\bar{A}} G \cos 2l - \frac{L}{\bar{A}} \left[\rho \cos\delta - \tau \cos(2l+\delta)\right] \\ \frac{dh}{dt} &= 0 \\ \frac{dJ}{dt} &= -\frac{\varepsilon}{\bar{A}} G \sin J \sin 2l - \frac{L}{\bar{A}} \sin J^\star \left[\rho \sin\delta + \tau \sin(2l+\delta)\right] \end{aligned} \tag{8}$$

where,

$$\rho \;=\; \frac{1}{2}\left[\alpha(1-\varepsilon)\,+\,\beta(1+\varepsilon)\right] \quad \text{and} \quad \tau=\frac{1}{2}\left[\alpha(1-\varepsilon)\,-\,\beta(1+\varepsilon)\right] \tag{9}$$

To integrate this system of first order differential equations, we have used a fifth-order adaptative stepsize Runge-Kutta-Fehlberg algorithm.

3. CONCLUSIONS AND APLICATIONS.

It should be worthy of noticing that this numerical approach provides not only a check on some classical results from the analytical methods but it is particulary useful and effective in obtaining the solution in a not so complicated way as that carried out using different approaches.

As the solution of this problem has a direct dependence with the principal moments of inertia of the body, the torque-free rotational motion will therefore provide an useful material relating to the behaviour of the body under a large-scale, and it may hence be of great interesting connection with the investigation of slow changes in the figure of the Earth and any other celestial bodies.

7. REFERENCES

Kinoshita, H.: 1977, "Theory of the rotation of the rigid Earth". *Celes. Mech.* **15**, 277-326.

Kubo, Y.: 1991, "Solution to the rotation of the elastic Earth by method of rigid dynamics". *Celes. Mech.* **50**, 165-187.

Lambeck, K.: 1980, "The Earth's variable rotation". *Cambridge University Press.*

Moritz, H. and Mueller, I.I.: 1987, "Earth rotation: Theory and Observation". *The Ungar Publishing Company*, New York.

Munk, W.H. and MacDonald, G.J.F.: 1975, "The rotation of the Earth: A geophysical discussion". *Cambridge University Press.*

Souchay, J., Folgueira, M. and Bouquillon, S.: 2002, "Effects of the triaxiality on the rotation of celestial bodies: application to the Earth, Mars and Eros". *(submitted to Earth, Moon and Planets).*

Souchay, J. and Folgueira, M.: 2002, "Solution to the torque-free rotational motion of an elastic body by a numerical integration method". *(in preparation).*