VARIATIONAL APPROACH TO THE ROTATIONAL DYNAMICS OF A THREE-LAYER EARTH MODEL: FLUID OUTER CORE INTERACTIONS

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ABSTRACT. By means of the Hamiltonian theory of the rotation of the non–rigid Earth, we have obtained explicit expressions of the torques exerted by the fluid on the solid layers of a two and three–layer Earth models as functions of the canonical Andoyer variables. When rewriting these formulae in terms of the components of angular velocities and symmetry axes of the layers, the transformated expressions are the same as those derived by other authors using different methods. Anyway, here the derivation is obtained in a much more simple way without the concurrence of Hydrodynamics equations.

1. INTRODUCTION

The determination of the rotational motion of a celestial body around its barycenter is one of the most important problems in Celestial Mechanics. This relevance is stressed if the celestial body is the Earth. The reason is clear: the accurate knowledge of Earth rotation is fundamental to tackle the definitions and realizations of space and time reference systems.

The Earth rotation problem can be studied by applying different approaches. One of them is based on the application of the Variational Principles of Mechanics, that is to say, the establishment and resolution of the problem is performed in the context of Lagrangian (Lagrange equations) or Hamiltonian (canonical equations) frameworks. At the beginning of the XXth century this line was followed by Poincaré (1910) and Andoyer (1923) when dealing with the rotational motion of non-rigid and rigid Earth models. Later, there have been other investigations sharing this approach such as the works of Kinoshita (1977), Moritz (1982), Getino and Ferrándiz (2001), etc. In the last years the rotational motion of a more sophisticated Earth model composed of three layers (mantle, fluid outer core and inner core) is also being investigated under this variational perspective by means of a Hamiltonian theory (e.g. Escapa et al. 2001) or with the help of Poincaré equations (Escapa et al. 2002).

A fact that must be underlined is that the resolution of the Earth rotation problem by means of the Variational Principles of Mechanics is not only interesting from an academic point of view but also from a practical one. Let us remember that the Hamiltonian theory of the rotation of the rigid Earth (Kinoshita 1977, Souchay et al. 1999) is probably the most complete and precise theory for this kind of Earth model available nowadays. Likewise, the Hamiltonian theory by Getino and Ferrándiz (2000) provides competitive rotational models for the non–rigid Earth. On the other hand, the variational approach presents some advantages with respect to other treatments based on the Vectorial Mechanics (e.g. Sasao et al. 1980, Mathews et al. 1991, 2002). In particular, the explicit computation of the torques exerted by the fluid on the solid layers is avoided.

In this note we focus our attention in obtaining the interactions among the fluid and the solid layers. Anyway, let us recall (Escapa et al. 2001, 2002) that in the variational context it is not necessary to know the explicit functional expression of these torques, since the equations of motion are derived from the Hamiltonian or Lagrangian of the system. However, it is interesting to have the expressions of the torques with a twofold aim: first, to gain some kinematical and geometrical insight into the interactional mechanism among the fluid and the solid layers. Second, to compare them with the expressions used in the vectorial approach which are obtained by means of a cumbersome procedure involving Hydrodynamics equations.

2. VECTORIAL MECHANICS APPROACH

The Vectorial Mechanics approach is based on the general equation of angular momentum conservation of a system

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N}, \tag{1}$$

where $\boldsymbol{L}$ is the angular momentum of the system and $\boldsymbol{N}$ the torque acting on it. Next, we sketch the basic features of the way in which eq. (1) is applied to solve Earth rotation problems. Specifically, we will consider the line followed by Sasao et al. (1980) and its generalizations. Anyway, let us point out that there are other approaches (e.g. Wahr 1981) starting basically from (1) that we will not treat in this note.

To model the rotational motion of the Earth it is considered one equation of the form (1) for each layer of the Earth. Besides it is necessary to perform some additional considerations that allows to tackle this complicated problem (see Kinoshita and Sasao 1989). One of the most important simplification is to assume that the field of velocities of each layer is composed of a dominant rigid–rotation term. So the angular momentum (rotational) of a layer has the form

$$\boldsymbol{L}_i = \Pi_i \boldsymbol{\varpi}_i, \tag{2}$$

being Π_i the tensor of inertia of the layer and $\boldsymbol{\varpi}_i$ its associated angular velocity (rigid–rotation term) with respect to an inertial frame. This one is decomposed as

$$\boldsymbol{\varpi}_i = \boldsymbol{\omega}_m + \boldsymbol{\omega}_i. \tag{3}$$

$\boldsymbol{\omega}_m$ is the associated angular velocity to the mantle. There are other possibilities to make this decomposition, see Mathews et al. 1991, but these ones do not change the fundamental idea of the method. In this way the equation (1) is written as

$$\frac{d\boldsymbol{L}_i}{dt} + \boldsymbol{\omega}_m \wedge \boldsymbol{L}_i = \boldsymbol{N}_{i\,out} + \boldsymbol{N}_{i\,int}. \tag{4}$$

The subscript i refers to the different layers (mantle, fluid outer core,...). We have described the evolution of $\boldsymbol{L}_i$ with respect to a frame (Tisserand frame) evolving with the angular velocity $\boldsymbol{\omega}_m$. In addition, we have split out the torque acting on the layer in two parts: $\boldsymbol{N}_{i\,out}$ is the torque due to the interactions produced outside the Earth and $\boldsymbol{N}_{i\,int}$ is the torque produced inside the Earth. The equation referring to the variation of the angular momentum of the mantle is substituted by one describing the behaviour of the whole Earth. Namely

$$\frac{d\boldsymbol{L}}{dt} + \boldsymbol{\omega}_m \wedge \boldsymbol{L} = \boldsymbol{N}_{out}, \tag{5}$$

with $\boldsymbol{L} = \sum \boldsymbol{L}_i$, $\boldsymbol{N}_{out} = \sum \boldsymbol{N}_{i\,out}$. To simplify the terminology we also refer to this equation as the equation of a layer (the whole Earth). The internal torques do not appear in the above formula because of Newton's action–reaction principle. Finally, to determine the rotational motion of the Earth we have to add to the dynamical eqs. (4) and (5) a new set of relations that involve the variables characterizing the Earth orientation, such as the case of the Euler angles. These relationships mix the time derivatives of Euler angles with the components of the angular velocities of each layer (see Escapa et al. 2002) and with (4) and (5) form the fundamental system of differential equations whose solutions provide the rotational motion of the Earth (precession, nutation and length of day).

In this framework it is possible to study the rotational motion of different Earth models. First, we have to fix the number of layers of our model, that is to say to consider a one–layer, two–layer or three–layer Earth model. Second, we have to provide analytical expressions for the quantities entering in eqs. (4) and (5). Depending on the physical characteristics of the model it will be necessary to consider different expressions for the tensors of inertia and for the external and internal torques. For example, the external torques could take into account the gravitational perturbations of moon, sun, etc. Some of the internal torques can be due to dissipative processes happening in the interior of the Earth, the interactions of the fluid with the solid layers (pressure torques), etc. Other characteristics of the Earth model, such as the elasticity of the layers, can also be fitted in this scheme by performing some approximations (Sasao et al. 1980).

The explicit expressions of the torques or tensor of inertia of each layer are derived following different methods. For instance, the torques of gravitational origin (internal and external) are obtained through a potential function; the dissipative torques are linear combinations of the components of the angular velocities of the layers (Sasao et al. 1980). In the case of the pressure torques the derivation of explicit expressions is more complicated. These expressions come from the equations of fluid motion following a cumbersome procedure (Sasao et al. 1980), specially in the case of a three–layer Earth model (Mathews et al. 1991).

The next step in the development of this approach would be to solve the differential equations of motion. The procedure followed is to use the so–called transfer function method, which essentially consists in taking advantage of the solution produced for a rigid Earth model. In this way it is not necessary to work with the explicit expressions of the gravitational potential of moon, sun and planets, although the method also presents some limitations (Escapa et al. 2002). This stage of the theory is out of the scope of this paper and will not be considered.

3. VARIATIONAL APPROACH

The Variational approach is based on consider the extremals of the variational problem

$$\delta \int_{t_1}^{t_2} F\,dt = 0, \tag{6}$$

where F is a function depending on the system. Starting from this standpoint several methods have been developed to treat the Earth rotation problem (Poincaré 1910, Moritz 1982, etc). The main approximations performed in the Vectorial approach (e.g. rigid–rotation field of velocities) are also assumed in these methods. We focus our attention on the Hamiltonian formalism due to Getino and Ferrándiz, giving a brief outline of the fundamentals of the method.

In this context the equations of motion of a system with n degrees of freedom are derived with the help of the Hamilton canonical equations

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + Q_{q_i}; \ \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} - Q_{p_i}; \ i = 1, \ldots, n. \tag{7}$$

p_i, q_i are the canonical variables, momenta and coordinates, that describe the dynamical behaviour of the system. H is the Hamiltonian function: it is a sum of the kinetic, T, and potential, V, energies of the system and Q_{ξ_i} are the generalized forces, necessary to model the dissipative or non-conservative processes. When studying the rotational motion of a system there are several possibilities to chose the canonical variables. One of the most useful is to employ the Andoyer canonical set (or some variation of this set, Getino 1995a, b, Escapa et al. 2001) because of its simplicity and the direct geometrical interpretation of its canonical momenta in terms of the components of the (rotational) angular momentum (Kinoshita 1977). In this way we associate one Andoyer set, composed of six canonical variables, to each layer of the Earth.

By so doing the equations of the rotational motion are established specifying the analytical expressions of T, V and Q_{ξ_i}, which depend on the physical characteristic of the Earth model. So, T is the sum of the (rotational) kinetic energy of each layer, this one can be computed through the equation

$$T_i = \frac{1}{2} < \Pi_i^{-1} \boldsymbol{L}_i, \boldsymbol{L}_i > . \tag{8}$$

$<,>$ stands for the scalar product in the real tridimensional space and Π_i^{-1} is the inverse of the tensor of inertia of the layer. V is the potential energy arising from the gravitational interactions of internal or external origin. This is derived from a potential function expressed in terms of the canonical variables by means of the Wigner's theorem. The generalized forces Q_{ξ_i} are obtained from the dissipative torques (González and Getino 1997).

To solve the equations of motion the Hamiltonian formalism exploits the powerful canonical perturbations methods. These ones present a great advantage with respect to the transfer function method, since the non–linear terms of the differential equations can be also studied with this formalism. This is not the case for the transfer function method, which is intrinsically a linear procedure. There are other advantages of the Hamiltonian formalism (Escapa et al. 2002) that we do not analyze in this work.

4. VARIATIONAL DERIVATION OF FLUID INTERACTIONS

In the previous sections we have described the way in which the different mechanics characterizing the Earth models are taken into account in the Vectorial and Hamiltonian approaches. Anyway, there is one interaction that is explicitly worked out by means of the Hydrodynamics equations in the Vectorial treatment and that, apparently, does not appear in the Hamiltonian formalism. We are referring to the fluid interaction (pressure torque). Where are this effect taken into account in the Hamiltonian formalism?

This effect is included in the Hamiltonian of the system through the kinetic energy of the fluid layer (Moritz 1982, Getino 1995a). In this way to derive the equations of motion of the system we do not need to employ the fluid motion equations to obtain its interactions; we have only to construct the kinetic energy of the system, which is a simple task. This is a great advantage of the Hamiltonian formalism, shared with other variational approaches, with respect to the Vectorial Mechanics method. Anyway, using the Hamiltonian formalism we can obtain the explicit expressions of these interactions. By so doing, we can compare these ones with the expressions employed in the Vectorial approach. With this aim we are going to compute the interaction fluid torques for a two and three–layer Earth models. Besides, due to the fact that these interactions are included through the kinetic energy we can consider simple Earth models where the solid layers are rigid and there is no gravitational interactions nor dissipative processes. This fact will simplify the exposition and the computations without changing the basic features of the method.

Two–layer Earth model

We will consider the free rotation of an Earth model composed of two layers: a rigid axial–symmetrical mantle that encloses a fluid core. The Hamiltonian of the system is

$$H = T_m + T_f = \frac{1}{2} < \Pi_m^{-1} \boldsymbol{L}_m, \boldsymbol{L}_m > + \frac{1}{2} < \Pi_f^{-1} \boldsymbol{L}_f, \boldsymbol{L}_f > . \tag{9}$$

To derive the form of the fluid interactions we write eq. (4) as

$$\boldsymbol{N}_{m\ int} = \frac{d\boldsymbol{L}_m}{dt} + \boldsymbol{\omega}_m \wedge \boldsymbol{L}_m. \tag{10}$$

In this situation $\boldsymbol{N}_{m\ int}$ is the torque due to the interaction of the fluid with the mantle ($\boldsymbol{N}_{m\ int} = -\boldsymbol{N}_{f\ int}$). To obtain the expression of the left hand side of eq. (10) we have to write the right hand side in terms of the elements of the Hamiltonian formalism, which are the known data. Besides, it is expedient to recall that the time derivative of a function of the canonical variables can be computed through the Poisson bracket

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} \right) + \frac{\partial f}{\partial t}, \tag{11}$$

being H the Hamiltonian of the system and p_i, q_i the canonical variables.

To compute the former formulae we have to specify a canonical set. As usual, we employ the Andoyer variables (Getino 1995a, b), in addition to take advantage of the geometrical meaning of this set we will express $\boldsymbol{\omega}_m$ and $\boldsymbol{L}_m$ in terms of the the angular momentum of the system $\boldsymbol{L}$ and of the fluid $\boldsymbol{L}_f$

$$\boldsymbol{L}_m = \boldsymbol{L} - \boldsymbol{L}_f ;\ \boldsymbol{\omega}_m = \Pi_m^{-1} \left(\boldsymbol{L} - \boldsymbol{L}_f \right). \tag{12}$$

So, the components of $\boldsymbol{N}_{m\ int}$ in the Tisserand frame will be given by

$$(N_{m\,int})_j = \left\{ (L)_j - (L_f)_j , H \right\} + \sum_{l,k,p=1}^{3} \varepsilon_{jlk} \left(\Pi_m^{-1} \right)_{lp} \left[(L)_p - (L_f)_p \right] \left[(L)_k - (L_f)_k \right], \ j = 1, 2, 3. \tag{13}$$

ε_{jkl} is the alternating symbol; j, k, l denotes the components in the Tisserand frame. Taking into account the relationships between the Andoyer variables with the components of the angular momentum and the form of the tensors of inertia of the mantle and the fluid (Getino 1995b), we obtain that the expression for the fluid interaction that turns out to be

$$\boldsymbol{N}_{m\ int} = -\boldsymbol{N}_{f\,int} = \boldsymbol{L}_f \wedge \left(\Pi_f^{-1} \boldsymbol{L}_f \right) = \boldsymbol{L}_f \wedge \left(\boldsymbol{\omega}_m + \boldsymbol{\omega}_f \right). \tag{14}$$

This expression is the same as that obtained by Sasao et al. (1980) by using the fluid motion equation. The same expression is also derived by Moritz (1982) using a variational approach based on Poincaré equations.

Three–layer Earth model

Next, let us consider a model composed of three layers: a rigid axial–symmetrical mantle, a fluid outer core and a rigid axial–symmetrical inner core. The Hamiltonian of the system that is equal to the kinetic energy of the layers is

$$H = T_m + T_f + T_s = \frac{1}{2} < \Pi_m^{-1} \boldsymbol{L}_m, \boldsymbol{L}_m > + \frac{1}{2} < \Pi_f^{-1} \boldsymbol{L}_f, \boldsymbol{L}_f > + \frac{1}{2} < \Pi_s^{-1} \boldsymbol{L}_s, \boldsymbol{L}_s >, \tag{15}$$

where the subscript s refers to the rigid inner core. For this model the fluid interacts both with the mantle and with the inner core, so we have to compute two torques. To do this we follow a similar procedure to the previous one considering the equations

$$\boldsymbol{N}_{m\,int} = \frac{d\boldsymbol{L}_m}{dt} + \boldsymbol{\omega}_m \wedge \boldsymbol{L}_m;\; \boldsymbol{N}_{s\,int} = \frac{d\boldsymbol{L}_s}{dt} + \boldsymbol{\omega}_m \wedge \boldsymbol{L}_s. \tag{16}$$

Therefore, the total torque acting on the fluid is $\boldsymbol{N}_{f\,int} = -\boldsymbol{N}_{m\,int} - \boldsymbol{N}_{s\,int}$. To write easily the former expressions in terms of the Andoyer set for a three–layer Earth model we will put

$$\boldsymbol{L}_m = \boldsymbol{L} - \boldsymbol{L}_f - \boldsymbol{L}_s;\; \boldsymbol{\omega}_m = \Pi_m^{-1}\left(\boldsymbol{L} - \boldsymbol{L}_f - \boldsymbol{L}\right). \tag{17}$$

In this way the components of the interaction torques in the Tisserand frame are

$$\begin{aligned}
(N_{m\,int})_j &= \Big\{(L)_j - (L_f)_j - (L_s)_j\,, H\Big\} + \\
&+ \sum_{l,k,p=1}^{3} \varepsilon_{jlk}\left(\Pi_m^{-1}\right)_{lp}\Big[(L)_p - (L_f)_p - (L_s)_p\Big]\left[(L)_k - (L_f)_k - (L_s)_k\right], \\
(N_{s\,int})_j &= \Big\{(L_s)_j\,, H\Big\} + \sum_{l,k,p=1}^{3} \varepsilon_{slk}\left(\Pi_m^{-1}\right)_{lp}\Big[(L)_p - (L_f)_p - (L_s)_p\Big](L_s)_k\,.
\end{aligned} \tag{18}$$

The right hand side of these equations are computed by expressing $\boldsymbol{L}$, $\boldsymbol{L}_f$, $\boldsymbol{L}_s$, Π_m, Π_f and Π_s in terms of the Andoyer canonical set (Escapa et al. 2001). After doing some algebra we obtain the torque acting on the fluid

$$\boldsymbol{N}_{m\ int} + \boldsymbol{N}_{s\,int} = -\boldsymbol{N}_{f\ int} = \boldsymbol{L}_f \wedge \left(\Pi_f^{-1}\boldsymbol{L}_f\right) = \boldsymbol{L}_f \wedge \left(\boldsymbol{\omega}_m + \boldsymbol{\omega}_f\right). \tag{19}$$

The expression related with the inner core is complicated. Anyway, if we only retain first order terms we get the simplified equations

$$\begin{aligned}
(N_{s\,int})_1 &= \Omega A_s \delta\left[(\omega_m)_2 + (\omega_f)_2 - (\Omega k_s)_2\right], \\
(N_{s\,int})_2 &= -\Omega A_s \delta\left[(\omega_m)_1 + (\omega_f)_1 - (\Omega k_s)_1\right], \\
(N_{s\,int})_3 &= 0.
\end{aligned} \tag{20}$$

$(k_s)_1$ and $(k_s)_2$ are the x and y components of the symmetry axis of the inner core on the Tisserand frame, Ω is the mean angular velocity of the Earth, A_s is the equatorial inertia moment of the inner core and δ is an adimensional parameter proportional to the dynamical ellipticity of the inner core (Escapa et al. 2001).

If we had employed the method used by Mathews et al. 1991 we would have obtained the same expressions as in (19) and (20). However, here the procedure has been much more simple, since we have only used the kinetic energy of the system. Finally, let us underline the fact that with the Hamiltonian formalism, or other variational method, it is not necessary to compute the expressions of the pressure torques to construct the equations of motion of the system, since these equations are directly derived from the Hamiltonian of the system. This is a great advantage with respect to the Vectorial Mechanics approaches.

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