TIME BASED ON EARTH ROTATION

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ABSTRACT. We examine the expression for the Earth's instantaneous angular velocity vector in terms of the Earth orientation parameters employed in the characterization of the transformation between the celestial and terrestrial reference frames, and confirm that the stellar angle, on which the definition of UT1 recently adopted by the IAU is based, is indeed the time integral of the component of the angular velocity vector in the direction of the Celestial Intermediate Pole (CIP). We consider also a different conceptual basis for a measure of time based on Earth rotation—a "World Time" (WT) which, unlike UT1, stays strictly in synchronism with the apparent motion of the Sun as seen from the Earth despite the precession of the Earth's axis, and is therefore most appealing as a time scale for civil use. Unlike UT1 based on the stellar angle, WT depends only on the instantaneous values of the Earth orientation parameters. It cannot, however, replace UT1, which is needed for the investigation of geophysical mechanisms responsible for variations in the Earth's rotation rate.

1. INTRODUCTION

The definition of UT1 is based on the angle through which the Earth rotates about a specified axis. Variability of the direction of the Earth's rotation axis, in space and relative to the Earth, makes it a nontrivial task to characterize precisely the most appropriate angle (about what axis? from what origin?), and to relate it to the Earth orientation parameters measured, e.g., by very long baseline interferometry (VLBI).

The definition adopted by the International Astronomical Union in 2000 relates UT1 to the stellar angle θ based on the concept of the "nonrotating origin" (Guinot, 1979) on the instantaneous equator associated with the celestial intermediate pole. In the decomposition of the transformation between the celestial and terrestrial reference frames (CRF and TRF) into polar motion, axial rotation about the CIP, and nutation-precession, θ appears in combination with s and s' of Capitaine et al. (1986), which are time integrals of quantities characterizing the paths of the CIP in the CRF and the TRF.

Possible alternative definitions of an origin have been considered and their relative merits discussed by Fukushima (2001).

However, the concept of an origin is not needed at all for expressing the Earth's angular velocity vector $\mathbf{\Omega}$ in terms of the celestial and terrestrial position coordinates (X, Y) and $(x_p, -y_p)$ of the CIP, the angle of rotation χ around the CIP, and the time derivatives of these quantities. One of our objectives here is to show explicitly, *without any reference to any origin*, that the

integral of the expression for the component of $\mathbf{\Omega}$ in the direction of the CIP does coincide with the stellar angle θ.

The use of an intermediate pole (the Celestial Ephemeris Pole or the recently adopted CIP) has been believed for long to be necessary for the estimation of the low frequency "polar motion" in the terrestrial frame. However, as has been shown by Mathews and Herring (2000), one could very well eliminate the intermediate pole and use the pole of the TRF itself as the reference pole (thereby making $x_p = y_p = 0$) and still estimate the polar motion from the high frequency part that would then be present in X and Y. The feasibility of such estimation has been demonstrated by Bizouard et al. (2000) from an analysis of simulated data. The stellar angle would then be simply the integral of the angular velocity component Ω_3 in the direction of the pole of the TRS which is essentially along the symmetry axis of the ellipsoidal Earth. The zonal tidal deformations and ocean tides which give rise to UT1 variations are symmetric about the third (polar) axis, and hence it is Ω_3 that is directly linked to UT1 variations. Therefore it would be most logical to use the angle θ defined as $\int \Omega_3 dt$, rather than the θ defined with reference to the CIP (a notional pole with no physical existence) as the basis for defining UT1.

A second objective of this paper is to introduce a different measure of time, also based on Earth rotation, which stays synchronized with the apparent motion of a suitably chosen celestial body in the sky (a fixed star/quasar), as viewed from the precessing, nutating, and rotating Earth. The "World Time" (WT) is obtained from the angle of rotation so defined, by taking account of the motion of the Sun relative to the fixed stars. WT would be strictly synchronous with the motion of the Sun in the sky, and a day of WT would be the interval between successive meridian crossings of the Sun. UT1, on the other hand, cannot remain in phase with the mean solar day on account of precession, as noted by Capitaine et al. (1986). It appears therefore that WT would be best suited for civilian use, while UT1 is undoubtedly the scale of choice for the study of geophysical phenomena affecting the angular velocity of Earth rotation.

2. TRANSFORMATION FROM CRF TO TRF, AND KINEMATIC RELATIONS

The transformation to the terrestrial reference frame from the space-fixed celestial one is

$$T = W^{-1}(x_p, y_p)\, R(\chi)\, S(X, Y), \tag{1}$$

where

$$W = \begin{pmatrix} 1 - x_p^2/2 & x_p y_p/2 & -x_p \\ x_p y_p/2 & 1 - y_p^2/2 & y_p \\ x_p & -y_p & 1 - (x_p^2 + y_p^2)/2 \end{pmatrix}, \qquad R = \begin{pmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

$$S = \begin{pmatrix} 1 - FX^2 & -FXY & -X \\ -FXY & 1 - FY^2 & -Y \\ X & Y & 1 - F(X^2 + Y^2) \end{pmatrix} \tag{3}$$

with $F = (1 + Z)^{-1}$, $Z = (1 - X^2 - Y^2)^{1/2}$. The matrix S transforms the celestial frame into one with its third axis in the direction of the CIP, R rotates the transformed frame around the CIP, and W^{-1} takes the third axis of the rotated frame into the third axis of the terrestrial frame.

The kinematic relations connecting the terrestrial components of the angular velocity vector $\mathbf{\Omega}$ to the Earth orientation parameters are encompassed in the matrix equation

$$M = \frac{dT}{dt} T^{-1} = \left(\frac{dW^{-1}}{dt} W\right) + W^{-1}\left(\frac{dR}{dt} R^{-1}\right) W + (W^{-1}R)\left(\frac{dS}{dt} S^{-1}\right)(W^{-1}R)^{-1}, \tag{4}$$

M is the antisymmetric matrix with elements

$$M_{2,3} = \Omega_1 \equiv \Omega_0 m_1, \quad M_{3,1} = \Omega_2 \equiv \Omega_0 m_2, \quad M_{1,2} = \Omega_3 \equiv \Omega_0(1+m_3). \tag{5}$$

On evaluating the right hand member of eq. (2), one obtains

$$\Omega_1 = L_{23} + L_{12}x_p - (1/2)(L_{23}x_p - L_{31}y_p)x_p - \dot{y}_p - (1/2)(x_p\dot{x}_p + y_p\dot{y}_p)y_p, \tag{6a}$$

$$\Omega_2 = L_{31} - L_{12}y_p + (1/2)(L_{23}x_p - L_{31}y_p)y_p - \dot{x}_p - (1/2)(x_p\dot{x}_p + y_p\dot{y}_p)x_p, \tag{6b}$$

$$\Omega_3 = L_{12}[1 - (1/2)(x_p^2 + y_p^2)] - (L_{23}x_p - L_{31}y_p) - (1/2)(y_p\dot{x}_p - x_p\dot{y}_p), \tag{6c}$$

where Ω_0 is the mean angular velocity of Earth rotation and

$$L_{23} = -L_{32} = (\dot{X} + \Delta_X)\sin\chi - (\dot{Y} + \Delta_Y)\cos\chi, \tag{7a}$$

$$L_{31} = -L_{13} = (\dot{X} + \Delta_X)\cos\chi + (\dot{Y} + \Delta_Y)\sin\chi, \tag{7b}$$

$$L_{12} = -L_{21} = \frac{d\chi}{dt} + \Delta_3, \tag{7c}$$

with

$$\Delta_X = X(X\dot{X} + Y\dot{Y}), \quad \Delta_Y = Y(X\dot{X} + Y\dot{Y}), \quad \Delta_3 = (Y\dot{X} - X\dot{Y})/(1+Z). \tag{8}$$

3. ANGULAR VELOCITY AND THE STELLAR ANGLE

How directly is the stellar angle related to the angular velocity?

Consider first the component $\mathbf{\Omega}\cdot\mathbf{e}$ of the angular velocity vector in the direction of the CIP. The unit vector $\mathbf{e}$ in this direction has components x_p, $-y_p$, $(1-x_p^2-y_p^2)^{1/2}$. Evaluation of $\mathbf{\Omega}\cdot\mathbf{e}$ with the help of eqs. (6) is then straightforward. One finds that a number of cancellations take place, leading to the end result:

$$\mathbf{\Omega}\cdot\mathbf{e} = \frac{d\chi}{dt} + \Delta_3 + \frac{1}{2}\left(y_p\frac{dx_p}{dt} - x_p\frac{dy_p}{dt}\right). \tag{9}$$

Integration of the above equation from an epoch t_0 up to t yields

$$\int_{t_0}^{t}\mathbf{\Omega}(t')\cdot\mathbf{e}(t')dt' = \chi(t) - \chi_0 + \int_{t_0}^{t}\Delta_3(t')dt' + \frac{1}{2}\int_{t_0}^{t}\left(y_p(t')\frac{dx_p(t')}{dt'} - x_p(t')\frac{dy_p(t')}{dt'}\right)dt'. \tag{10}$$

On taking note of the expression given in eq. (8) for Δ_3, one recognizes readily that the last two integrals in (10) are the quantities s and $-s'$, respectively, of Capitaine (1990). Thus

$$\int_{t_0}^{t}\mathbf{\Omega}(t')\cdot\mathbf{e}(t')dt' = \chi(t) - \chi_0 + s - s'. \tag{11}$$

To see the connection of this result with the stellar angle θ, observe that in the notation of Capitaine, the transformation from the CRF to the TRF is

$$T = W(t)R(t)NP(t), \tag{12}$$

where the quantities in (12) go over to the following quantities in (1):

$$W(t)R_3(-s') \to W^{-1}(x_p, y_p), \quad R_3(s')R(t)R_3(-c_0 - s) \to R(\chi), \quad R_3(c_0+s)NP(t) \to S(X,Y), \tag{13}$$

where $R_i(\alpha)$ is a rotation about the ith axis through the angle α. Since $R(t)$ is $R_3(\theta)$ and our $R(\chi)$ is simply $R_3(\chi)$, it follows that

$$\chi = \theta - (s + c_0) + s'. \tag{14}$$

On comparing this result with eq. (11), we obtain explicit confirmation that the stellar angle θ is the time integral of the component of the angular velocity vector in the direction of the CIP:

$$\theta(t) = \int_{t_0}^{t} \mathbf{\Omega}(t') \cdot \mathbf{e}(t') dt' + \chi_0 + c_0. \tag{15}$$

If the pole of the TRF were used as the reference pole as proposed by Mathews and Herring (2000), eliminating the intermediate pole (CIP), then x_p and y_p would vanish, W in eq. (4) would reduce to the unit matrix, and all terms in eqs. (6) involving x_p or y_p (or either of their derivatives) would drop out. The relation (15) would simply get replaced by

$$\theta(t) = \int_{t_0}^{t} \Omega_3(t') dt' + \chi_0 + c_0. \tag{16}$$

The derivation of this result would go through exactly as before, but with the simplifications mentioned above. It is to be noted, however (Mathews and Herring, 2000) that parts with diurnal and higher frequencies will now be present in X and Y:

$$X(t) = X_0(t) + \sum_{n \neq 0} [X_n(t) \cos(n\Omega_0 t) - Y_n \sin(n\Omega_0 t)] \tag{17a},$$

$$Y(t) = Y_0(t) + \sum_{n \neq 0} [X_n(t) \sin(n\Omega_0 t) + Y_n \cos(n\Omega_0 t), \tag{17b}$$

with the spectrum of every $X_n(t)$ and $Y_n(t)$, $(n \neq 0)$, confined to low frequencies $(-\Omega_0/2$ to $\Omega_0/2)$. The roles earlier played by x_p and y_p will be mimicked now by X_1 an Y_1.

The real distinction between (15) and (16) consists in the physical significance of the angle θ. In the former case, it is the cumulative angle of rotation about an axis which has no physical existence. (In fact, inasmuch as its definition depends in an essential manner on a separation between low and high frequency components of its motion, the CIP is nonlocal in the time variable.) On the other hand, θ in (16) is the rotation about the Earth's symmetry axis, which is indeed the axis of reference for the zonal tides which give rise to UT1 variations. Therefore, it would be most logical, in principle, to use this θ in defining UT1. However, there is no possibility, at present, of computing the stellar angle defined through (16) because the procedure envisaged in the work of Mathews and Herring (2000) for representing the spectral content of the Earth orientation parameters in different spectral bands is yet to be implemented in software for the analysis of space geodetic data.

4. TIME SYNCHRONIZED TO THE MOVEMENT OF STARS IN THE SKY

For the definition of such a time scale, one needs to specify a reference star. Rather than select a particular physical object, we represent the direction of the "reference star" by the point Σ_0 in the direction of x-axis of the currently adopted CRF. Define a sidereal time by the Greenwich hour angle (GHA) of this space-fixed point. The unit vector in the direction of Σ_0 has the coordinates $(T_{1,2}, T_{1,2}, T_{1,3})$ in the terrestrial frame. So the GHA of Σ_0 is

$$(GHA)_{\Sigma_0} = -\text{arc tan}(T_{1,2}/T_{1,1}). \tag{18}$$

This angle provides a measure of time, $(GST)_{\Sigma_0}$, that directly reflects the positions of stars in the sky.

Since the transformation matrix T is expressed in terms of the EOP through eq. (1), this time measure is in terms of the *instantaneous* values of the EOP. No integration over time is involved, unlike in the case of UT1.

The solar time WT will be related to $(GHA)_{\Sigma_0}$ in the same manner as UT1 is related to the stellar angle θ.

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