INFLUENCE OF THE TRIAXIALITY OF THE NON-RIGID EARTH ON THE J_2 FORCED NUTATIONS

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ABSTRACT. This note deals with the effects of the triaxiality of a two-layer Earth model on the nutations caused by the zonal harmonic of the second degree J_2. In particular, applying a Hamiltonian formalism, we calculate the analytical expressions of the forced nutations when considering the triaxiality in the kinetic energy of the model. The most relevant fact is the appareance of semidiurnal nutations. We evaluate their amplitudes, discussing their dependence as a function of the triaxiality of the fluid core.

1. INTRODUCTION

Usually, Hamiltonian studies of the rotation of the rigid and non-rigid Earth (for example, Souchay *et al.*, 1999; Getino, Ferrándiz and Escapa, 2001) include the triaxiality of the Earth solely by incorporating the non-zonal harmonics in the geopotential expansion. These terms give rise to the so-called short period nutations (diurnal and subdiurnal), which reach amplitudes at the level of a few microarcseconds. This kind of influence is called direct effect of the triaxiality (Escapa, Getino and Ferrándiz, 2001). However, the triaxiality of the Earth also modifies the kinetic energy of the system and, therefore, changes the analytical expression of the unperturbed, or zero order, Hamiltonian. So, when performing a canonical perturbation method, which depends on the differential equations generated by the unperturbed Hamiltonian, all the terms of the geopotential (zonal and non-zonal terms) are indirectly affected by the triaxiality. This effect is called indirect effect of the triaxiality.

In the case of the Hamiltonian theory for the rigid Earth, the indirect effect has been worked out by Escapa, Getino and Ferrándiz (2001) by expressing the potential energy in terms of the action-angle variables of the system. Here, we study the influence of this effect when considering a two-layer Earth model. In this way, we complete a set of works (González and Getino, 1997; Getino, González and Escapa, 2000; Getino, Ferrándiz and Escapa, 2001) in which the triaxiality of the non-rigid Earth is investigated by means of a Hamiltonian approach. As far we know, this is the only approach dealing with the two kinds of effects of the triaxiality (direct and indirect) of the non-rigid Earth.

2. HAMILTONIAN FOR THE PROBLEM

In this note, we will consider an Earth model composed by two layers: a non-symmetrical rigid mantle which encloses a fluid core. Both layers share a common barycenter. The kinetic energy of the system is given (González and Getino, 1997) by

$$2T = (\mathbf{L} - \mathbf{L}_c)^t \mathbf{\Pi}_m^{-1} (\mathbf{L} - \mathbf{L}_c) + \mathbf{L}_c \mathbf{\Pi}_c^{-1} \mathbf{L}_c. \tag{1}$$

$\mathbf{L}$ is the total angular momentum of the Earth, $\mathbf{L_c}$ is the angular momentum of the fluid core. In the principal frame of the mantle the inertia tensors of the mantle and of the core have the expressions

$$\mathbf{\Pi}_m = \begin{pmatrix} A - A_c & 0 & 0 \\ 0 & B - B_c & 0 \\ 0 & 0 & C - C_c \end{pmatrix}, \mathbf{\Pi}_c = \begin{pmatrix} A_c & 0 & 0 \\ 0 & A_c & 0 \\ 0 & 0 & C_c \end{pmatrix}, \tag{2}$$

being A, B and C the principal moments of inertia of the Earth and A_c, B_c and C_c the principal moments of inertia of the fluid core. To facilitate the computations, it is useful to introduce the adimensional parameters

$$e = \frac{C - A}{A},\ e_c = \frac{C_c - A_c}{A_c},\ d = \frac{1}{2}\left(1 - \frac{A}{B}\right),\ d_c = \frac{1}{2}\left(1 - \frac{A_c}{B_c}\right),\ r_{cm} = \frac{A_c}{A_m}. \tag{3}$$

e and e_c are the dynamical ellipticities of the Earth and of the core; d and d_c are related with the triaxiality of the model. From now on, we call them triaxiality parameters. It is worthy to note that for a symmetrical model the equatorial moments of inertia of the layer are equal and, therefore, the triaxiality parameters are equal to 0. In the case of the Earth, e, e_c, d and d_c are small quantities, hence, in the calculations we only retain first order terms with respect to these parameters. So, the kinetic energy of the model can be written as

$$T = T_s + \Delta_{Tr} T, \tag{4}$$

being T_s the kinetic energy corresponding to the symmetrical case and $\Delta_{Tr} T$ the contribution due to the triaxiality of the model, which, accordingly to the previous comments, will be proportional to d and d_c. This splitting of the kinetic energy will help us to isolate the influence of the non-symmetry on the nutational motion of the model.

Next, to construct a Hamiltonian theory, it is necessary to express the kinetic energy as a function of a canonical set. With this aim, we employ the canonical variables of Andoyer: M, N, Λ, M_c, N_c, Λ_c, μ, ν, λ, μ_c, ν_c, λ_c, (Getino, 1995). In terms of this set, the components of the angular momenta in the mantle frame are written as

$$\mathbf{L} = (M \sin\sigma \sin\nu, M \sin\sigma \cos\nu, N)^t,\ \mathbf{L}_c = (M_c \sin\sigma_c \sin\nu_c, -M_c \sin\sigma_c \cos\nu_c, N_c)^t, \tag{5}$$

with $N = M \cos\sigma$, $N_c = M_c \cos\sigma_c$. Hence, the kinetic energy is a function of the canonical set of the form

$$T = T_s(M, M_c, N, N_c, \nu, \nu_c) + \Delta_{Tr} T(M, M_c, N, N_c, \nu, \nu_c). \tag{6}$$

With regard to the potential energy of the system, arising from the gravitational interaction with the Moon and the Sun, let us remember that its analytical expresion is given by a sum of spherical harmonics (Kinoshita, 1977), known as geopotential expansion. In this expansion there are two kinds of terms: the zonal terms and the non-zonal terms. The last ones arise when the perturbed body is non-symmetrical. In this way, the potential energy of the system can also be written as

$$V = V_s\,(M, N, \Lambda, \mu, \lambda) + \Delta_{Tr} V\,(M, N, \Lambda, \mu, \nu, \lambda). \tag{7}$$

The explicit expression of the different parts of the geopotential in terms of the Andoyer canonical variables is worked out in detail in Kinoshita (1977).

From eqs.(6) y (7), it is possible to construct the Hamiltonian of the system. It turns out to be

$$H = T(M, M_c, N, N_c, \nu, \nu_c) + V(M, N, \Lambda, \mu, \lambda). \tag{8}$$

Strictly speaking, we should add to this expression a term due to the movement of the ecliptic. Anyway, this term only affects, at the first order, the preccesional motion. Because in this work we are focused on the nutational motion, we can disregard it without harm.

3. FIRST ORDER INTEGRATION

To solve analitically the equations of motion generated by the Hamiltonian H, we take advantage of the Horis's canonical perturbation method (Hori, 1966). To this end, we decompose the Hamiltonian as a sum of the form

$$H = H_0 + H_1 + ..., \text{ with } H_0 = T_s + \Delta_{Tr}T,\ H_1 = (V_2)_s + \Delta_{Tr}(V_2). \tag{9}$$

V_2 is the second degree part of the geopotential. It gives the main contribution to the potential energy of the system. The former decomposition verifies that the zero order part of the Hamiltonian is biggest than the first order part. We can achive a more simplified form by considering that, in this work, we are investigating the influence of the triaxiality part of the kinetic energy on the nutations. Since, the triaxiality of the Earth is small, the combined effect of the triaxiality of the Earth on the kinetic energy, $\Delta_{Tr}T$, and on the potential energy, $\Delta_{Tr}(V_2)$, will be very small. Due to this fact, we will neglect the term $\Delta_{Tr}(V_2)$ in H_1.

The first step to apply the Hori's method is to find a solution of the differential equations generated by H_0. This one turns out to be

$$M = c_1,\ \Lambda = c_2,\ \lambda = c_3,\ M_c = c_4,\ \Lambda_c = c_5,\ \lambda_c = c_6, \tag{10}$$

where c_i are integration constants. The remaining solutions are derived with the help of the system

$$\frac{d}{dt}\begin{pmatrix} iM\sin\sigma e^{-i\nu} \\ -iM_c\sin\sigma_c e^{i\nu_c} \\ -iM\sin\sigma e^{i\nu} \\ iM_c\sin\sigma_c e^{-i\nu_c} \end{pmatrix} = i\Omega\mathbf{J}\begin{pmatrix} iM\sin\sigma e^{-i\nu} \\ -iM_c\sin\sigma_c e^{i\nu_c} \\ -iM\sin\sigma e^{i\nu} \\ iM_c\sin\sigma_c e^{-i\nu_c} \end{pmatrix}. \tag{11}$$

being $\mathbf{J} = (\mathbf{J})_s + \Delta_{Tr}(\mathbf{J})$. The first part of this matrix is the same as the obtained for the symmetrical case. The last one is due to the triaxiality of the model. The eigenvalues of $\mathbf{J}$ are given, at first order in e, e_c, d and d_c, by

$$m_1 = (1 + r_{cm})\sqrt{e^2 - 2ed},\ m_2 = -[1 + (1 + r_{cm})(e_c - d_c)],\ -m_1,\ -m_2. \tag{12}$$

$\pm m_1$ are related with the Chandler wobble, whereas $\pm m_2$ are related with the Free Core Nutation. It is worthy to note that the triaxiality modifies the analitycal expressions of both modes with respect to the symmetrical case. Eqs.(12) are equivalent to those ones obtained by González and Getino (1997).

Once we have found the solutions of the zero order part, the analytical expressions of the periodic part of the motion (forced nutations) are computed through the generating function, W. In our case, it is given by

$$W = \int H_{1\,per}\,dt = \int (V_2)_{s\,per}\,dt, \tag{13}$$

where the subscript *per* notes the periodic part. For our purposes, this term can be described properly by the expression (Getino and Ferrándiz, 2001)

$$(V_2)_{s\,per} = k' \sum_{i \neq 0} B_i \cos \Theta_i - k' \sin \sigma \sum_{i,\tau=\pm 1} C_i(\tau) \cos(\mu - \tau \Theta_i) . \tag{14}$$

k' is a numerical constant depending on the perturbing body (the Moon or the Sun in this case). B_i y $C_i(\tau)$ are functions of I ($\Lambda = M \cos I$), and Θ_i depends on λ and on the Delaunay variables of the Sun and the Moon. A precise definition of these functions is given in Kinoshita (1977).

The computation of the integral (13) along the solutions (10) and (11) is involved. Anyway, by performing a procedure similar to the described in Getino and Ferrándiz (2001), we have obtained

$$\begin{aligned} W = W_1 + W_2 + W_3 + \ldots = k' \sum_{i \neq 0} \frac{B_i}{n_i} \cos \Theta_i - k' \sin \sigma \sum_{i,\tau=\pm 1} C_i(\tau) R_{11} \cos(\mu - \tau \Theta_i) + \\ + k' \sin \sigma \sum_{i,\tau=\pm 1} C_i(\tau) R_{13} \cos(\mu + 2\nu - \tau \Theta_i) + \ldots . \end{aligned} \tag{15}$$

R_{ij} denotes the ij element of the matrix $[(\Omega - \tau n_i)\,\mathbf{1} + \Omega \mathbf{J}]^{-1}$, being Ω the mean angular velocity of the Earth and $n_i = d\Theta_i/dt$.

As it can be seen, the generating function is composed of three relevants parts: W_1, W_2 and W_3. With respect to the first part, let us point out that the expression of W_1 is not affected by the triaxiality of the model. Moreover, this term is exactly the same that the obtained in Getino (1995). It produces the so-called Poisson terms of the long-period nutations. The second part W_2 is slightly affected by the triaxiality through the matrix $\mathbf{J}$. Anyway, its functional dependence is the same as the corresponding part of the generating function of the symmetrical case (Getino, 1995). It gives rise to the Oppolzer terms of the long-period nutations. The third part, W_3, is completely new, and it is enterely due to the triaxiality of the model. We will show below that it gives rise to forced nutations of short period. We will focus our attention in this part of the generating function.

4. SEMIDIURNAL TERMS: NUMERICAL REPRESENTATION

Let us remember that the periodic variation, at the first order, of any function of the canonical variables is given by $\Delta f = \{f, W\}$, where $\{,\}$ denotes the Poisson bracket of the system. In particular, the longitude of the node, $-\epsilon_f$, and the obliquity, $-\psi_f$, of the plane perpendicular to the figure axis can be written in terms of the Andoyer variables (Kinoshita, 1977) as

$$\psi_f = -\lambda - \frac{\sigma}{\sin I} \sin \mu, \;\; \epsilon_f = -I - \sigma \cos \mu. \tag{16}$$

So, by computing the Poisson bracket of these equations with $W = W_3$ we can derive that

$$\begin{aligned} \Delta \epsilon_f &= -k \sum_{i,\tau} \left\{ [C_i(\tau) R_{13} \cos g] \cos(2\phi - \tau \Theta_i) - [C_i(\tau) R_{13} \sin g] \sin(2\phi - \tau \Theta_i) \right\}, \\ \Delta \psi_f &= -\frac{k}{\sin I} \sum_{i,\tau} \left\{ [C_i(\tau) R_{13} \cos g] \sin(2\phi - \tau \Theta_i) + [C_i(\tau) R_{13} \sin g] \cos(2\phi - \tau \Theta_i) \right\}. \end{aligned} \tag{17}$$

$\phi = \mu + \nu - g/2$ is the sidereal time, which is referred to the Greenwich meridian and $k'/M = k$. The explicit expression of R_{13} is given by

$$R_{13} = \frac{\Omega(1 + r_{cm}) \left[d_c r_{cm} (\tau n_i - \Omega)^2 - d(1 + r_{cm})(\tau n_i - 2\Omega) \tau n_i \right]}{(\Omega - \tau n_i - \Omega m_1)(\Omega - \tau n_i + \Omega m_1)(\Omega - \tau n_i - \Omega m_2)(\Omega - \tau n_i + \Omega m_2)}. \tag{18}$$

The nutations given by eq. (17) are short period nutations, or semidiurnal, because the argument 2ϕ-$\tau\Theta_i$ has a period of about a half of a sidereal day. Besides, it is interesting to note that, unlike the semidiurnal nutations coming from the non-zonal parts of the geopotential of the second degree (Getino, Ferrándiz and Escapa, 2001), the amplitude of the nutations presents a resonance when n_i is equal to the Free Core Nutation. That is to say, this effect is strongly dependent on the Earth model considered.

To evaluate numerically the amplitudes of the nutations we will employ the following values for the parameters appearing in (17): $k_{moon} = 7567.870647''$/Jcy, $k_{sun} = 3474.613746''$/Jcy, $CW = 400.7$ sidereal days, $FCN = 432.94$ sidereal days (Getino and Ferrándiz, 2001); $I = -.4090926298$, $\Omega = 230121.6483$ rd/Jcy (IERS, 1996); $g/2 = -.2605521212$ rd, $\bar{C}_{22} = 2.4386 \times 10^{-6}$, $\bar{S}_{22} = -1.400 \times 10^{-6}$ (Bretagnon *et al.*, 1997); $(\bar{C}_{22})_c = 2.400 \times 10^{-6}$, $(\bar{S}_{22})_c = -1.191 \times 10^{-6}$ (González and Getino, 1997); $A = 8.0115 \times 10^{37}\,\mathrm{kg{\cdot}m^2}$, $A_c = 9.1168 \times 10^{36}\,\mathrm{kg{\cdot}m^2}$, $M = 5.9732 \times 10^{24}\,\mathrm{kg}$, $M_c = 1.9395 \times 10^{24}\,\mathrm{kg}$ (PREM, taken from Mathews *et al.*, 1991). The moment of inertia B is obtained through the non-zonal harmonics of the second degree, by means of the relationship

$$B = A + 4Ma^2\sqrt{\frac{5}{12}}\sqrt{(\bar{C}_{22})^2 + (\bar{S}_{22})^2}. \tag{19}$$

An analagous expression is valid for the core. From these considerations we get that $d = 1.09856 \times 10^{-5}$, $d_c = 8.91125 \times 10^{-6}$, $m_1 = 2.4956 \times 10^{-3}$ and $m_2 = -1.002309789$.

By substituting the former numerical values in (17), we determine the amplitudes of the forced nutations due to the inclusion of the triaxiality in the kinetic energy. The results are presented below, where only the terms greater than .1 μas (microarcsecond) have been displayed.

Table I: Figure axis nutations (Unit=μas)

Period	Alias	Longitude				Obliquity			
(days)	(days)	sin		cos		sin		cos	
		IT	NZ	IT	NZ	IT	NZ	IT	NZ
.498634	∞	-5.304	35.944	3.044	-20.632	1.211	-8.207	2.110	-14.297
.498598	-6798.36	-.765	4.872	.439	-2.796	.175	-1.112	.304	-1.938
.500000	182.62	.585	-11.987	-.336	6.881	-.134	2.737	-.233	4.768
.517526	13.66	.333	-28.622	-.191	16.430	-.076	6.535	-.132	11.385
.498671	6798.36	.099	-.712	-.057	.408	-.023	.162	-.039	.283
.497955	-365.26	.215	.281	-.123	-.161	-.049	-.064	-.085	-.112

In Table I, IT notes the values obtained in this work, whereas NZ denotes the values obtained by Getino, Ferrándiz and Escapa (2001) when evaluating the semidiurnal nutations of a two-layer Earth model coming from the non-zonal harmonics of the second degree. As it can be seen, the amplitude of the obtained nutations obtained are small. Anyway, some of them reach a few microarcseconds. It is also important to note that, due to the resonance, for the argument alias -365.26 the value of the amplitude is of the same order of magnitude as the value corresponding to NZ.

Finally, it is interesting to evaluate the variation of the amplitudes with respect to d_c. Let us point out, that there is a great uncertainty in the value of this parameter (see, for example, Brzezinski and Capitane, 2001). With this aim, and as a qualitative example, we have recomputed the amplitudes for the values $d_c = a = 1.7823 \times 10^{-6}$, $d_c = b = 8.91125 \times 10^{-6}$ and $d_c = c = 4.4556 \times 10^{-5}$, assuming that the other parameters remain constant. In the next table, we show the amplitudes obtained for the sin component of the nutation in longitude of the figure axis.

Table II: Variation with d_c (Unit=μas)

Argument						Period	Longitude (sin)			Alias
ϕ	l_M	l_S	F	D	Ω	(days)	a	b	c	(days)
2	0	0	0	0	0	.498634	-1.061	-5.304	-26.519	∞
2	0	0	0	0	-1	.498598	-.151	-.765	-3.836	-6798.36
2	0	0	-2	2	-2	.500000	.167	.585	2.677	182.62
2	0	0	-2	0	-2	.517526	.237	.333	.813	13.66
2	0	0	0	0	1	.498671	.020	.099	.494	6798.36
2	0	-1	0	0	0	.497955	.032	.215	1.127	-365.26

As it can be checked in Table II, a variation in the value of d_c affects almost linearly to the nutation amplitudes, in this sense, it would be important to determine with greater accuracy the value of d_c. On the other hand, we can see that in the case c the amplitude of the argument alias -365.26 is much greater than the obtained one from NZ (Table I). This fact might provide some help to estimate d_c from observational data.

5. ACKNOWLEDGMENTS

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