On solution of secular system in the analytical Moon's theory

Ivanova Tamara
Institute of applied astronomy of RAS
itv@ipa.nw.ru

ABSTRACT. This paper is the final step to construct the analytical theory of the orbital Moon's motion with taking into account the planetary perturbations in the form of the general planetary theory (GPT). Trigonometric theory of the secular perturbations in the motion of the major planets is used to obtain a trigonometric solution of the secular system for the Moon's motion. This solution has been built up to the tenth degree terms with respect to the eccentricities and inclinations of all the bodies. The right-hand members of the secular system for the Moon were obtained earlier in the purely analytical form but the trigonometric solution of the secular system has the semi-analytical form. It includes two basic sets of inequalities due to the secular evolution of the lunar perigee and node as well as of that of the major planets. All the analytical calculations are performed by the echeloned Poisson series processor EPSP.

1. INTRODUCTION

The theory of the orbital motion of the Moon in the frameworks of the general planetary theory GPT (Brumberg, 1995) enables to represent the lunar coordinates in a purely trigonometric form and valid, at least formally, for an indefinite interval of time. For that the Moon is considered to be an additional planet in the field of eight major planets. The trigonometric form of the coordinates is ensured by a special technique of the solution of the lunar equations that enables to separate the short–period and long–period terms arguments. The long–period terms form an autonomous secular system. The trigonometric solution of this system describes the secular motions of the lunar perigee and node with taking into account the secular planetary inequalities. The secular system in Laplace–type variables was constructed in (Ivanova, 2011). The aim of this paper is to solve this secular system by the normalizing Birkhoff transformation.

2. BASIC NOTIONS

Let us recall the basic notions of the construction of the Moon's theory (in detail in Ivanova, 2011). The differential equations of the Moon's motion in terms of the variables p, q and w have form

$$\ddot p + 2\sqrt{-1}n\dot p - \frac{3}{2}n^2(p+q) = n^2P\,,$$

$$\ddot w + n^2 w = n^2 W \qquad (1)$$

with the right–hand members

$$P = -1 - \frac{1}{2}p - \frac{3}{2}q + \frac{2}{n^2a^2}\frac{\partial U}{\partial \bar q}\,, \qquad W = w + \frac{1}{n^2a^2}\frac{\partial U}{\partial \bar w}\,, \qquad (2)$$

U being the force function in the form

$$U = n^2a^2\Bigg[\frac{a}{r} + \left(\frac{n_3}{n}\right)^2\frac{M_s}{M_s+M_3}\sum_{k=1}^{\infty}A_3^{(k-1)}\left(\frac{r}{a}\right)^{k+1}\left(\frac{a_3}{r_3}\right)^{k+2}P_{k+1}(\omega_3)$$

$$+\sum_{i=1,i\neq3}^{8}\left(\frac{n_i}{n}\right)^2\frac{M_i}{M_s+M_i}\sum_{k=1}^{\infty}A_i^{(k-1)}\left(\frac{r}{a}\right)^{k+1}\left(\frac{a_i}{\Delta_{3i}}\right)^{k+2}P_{k+1}(\omega_i)\Bigg]. \qquad (3)$$

The dimensionless complex conjugate variables p, q and real variable w representing small deviations from the planar circular motion are introduced instead of geocentric lunar rectangular coordinates $\mathbf{r} = (x, y, z)$

$$x + \sqrt{-1}y = a(1-p)\exp\sqrt{-1}\lambda, \quad q = \bar p, \quad z = a\,w, \quad w = \bar w. \qquad (4)$$

The heliocentric coordinates $\mathbf{r}_i = (x_i, y_i, z_i)$ of the principal planets are subjected to the similar transformation

$$x_i + \sqrt{-1}y_i = a_i(1-p_i)\exp\sqrt{-1}\lambda_i, \quad q_i = \bar{p_i}, \quad z_i = a_i w_i, \quad w_i = \bar{w_i}. \qquad (5)$$

Here the bar marks a conjugate quantity. The index i points to the principal planet with number i ($i = 1, 2, \ldots, 8$). For the Moon, the values both with index 9 and without any indices are used.
a, n, $\mathbf{r}$, λ are the semi-major axis, mean motion, radius–vector and mean longitude of the Moon, respectively.
a_i, n_i, $\mathbf{r}_i$, λ_i ($i = 1, 2, \ldots, 8$) are the same for the major planets.
M_s, M_e, M_m, M_i ($i = 1, 2, \ldots, 8$) are the masses of the Sun, the Earth, the Moon and the planets, respectively.
$P_k(\omega_i)$ are the Legendre polynomials.

3. INTERMEDIARY

The terms with $m = 0$ in (6) represent an intermediate solution independent of eccentricities and inclinations of all the bodies. It is sought by iterations. In the previous papers (for instance, Ivanova, 2013) it had the form of the Poisson series due to the expansion of the frequency denominators with respect to $\frac{n_i}{n}$ ($i = 1, 2, \ldots, 8$) obtained in the result of the integration of the intermediary equations. In this paper the intermediate solution has the form of the echeloned series without any expansion. The initial terms of the intermediary in the echeloned form are represented as

$$p^{(0)} = \sum_{i=1}^{8}M^{(i)}\Bigg\{\left(\frac{n_i}{n}\right)^2\Bigg[\frac{1}{6}B_i^3 + \frac{B_i^5}{(2\frac{n_i}{n}-3)(\frac{n_i}{n}-1)^2(2\frac{n_i}{n}-1)}\left(\frac{57}{16}\zeta_i^{-2} - \frac{9}{16}\zeta_i^2\right) -$$

$$-\frac{B_i^4G_i}{(\frac{n_i}{n}+\frac{n_3}{n}-3)(\frac{n_i}{n}+\frac{n_3}{n}-2)^2(\frac{n_i}{n}+\frac{n_3}{n}-1)}\left(\frac{57}{2}\zeta_i^{-1}\zeta_3^{-1} - \frac{9}{2}\zeta_i\zeta_3\right)\Bigg] + \ldots\Bigg\},$$

$\zeta_i = \exp\sqrt{-1}(\lambda - \lambda_i)$, $i = 1, 2, \ldots, 8$,

$$M^{(3)} = \frac{M_s}{M_s + M_3}, \; B_3 = 1, \; G_3 = 1,$$

$$M^{(i)} = \frac{M_i}{M_s + M_i}, \; B_i = \frac{a_i}{\sqrt{a_3^2 + a_i^2}}, \; G_i = \frac{a_3}{\sqrt{a_3^2 + a_i^2}}, \; i = 1, 2, 4, \ldots, 8.$$

The basic series of the Moon's theory have the form

$$p = \sum_{m=0}^{\infty}\sum_{i+j+k+l=m}p_{i,j,k,l}(t)\prod_{n=1}^{9}a_n^{i_n}\bar a_n^{j_n}b_n^{k_n}\bar b_n^{m_n}\,, \qquad (6)$$

$$w = \sum_{m=1}^{\infty}\sum_{i+j+k+l=m}w_{i,j,k,l}(t)\prod_{n=1}^{9}a_n^{i_n}\bar a_n^{j_n}b_n^{k_n}\bar b_n^{m_n} \qquad (7)$$

where a_n, b_n are the complex Laplace–type variables proportional to the eccentricity and inclination of the body with number n. The coefficients in (6) and (7) are quasi-periodic functions of mean longitudes of the major planets and the Moon

$$p_{i,j,k,l}(t) = \sum_{\gamma}p_{i,j,k,l,\gamma}\exp\sqrt{-1}\,(\gamma\lambda)$$

$$w_{i,j,k,l}(t) = \sum_{\gamma}w_{i,j,k,l,\gamma}\exp\sqrt{-1}\,(\gamma\lambda) \qquad (8),$$

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_9), \quad (\gamma\lambda) = \sum_{i=1}^{9}\gamma_i\lambda_i, \quad \sum_{i=1}^{9}\gamma_i = 0.$$

The coefficients $p_{i,j,k,l,\gamma}$ and $w_{i,j,k,l,\gamma}$ are the functions of the semi-major axes, mean motions and masses of the bodies under consideration.

It should be noted that the planetary theory is limited by the keplerian terms, the first–order with respect to the masses intermediary and linear theory. It is sufficient for constructing the theory of the Moon.

4. SECULAR SYSTEM

The terms with $m > 0$ in (6) and (7) are obtained by integrating the equations (1) and (2), respectively, for non–zero values of the eccentricities and inclinations of all the bodies. The technique of Birkhoff normalization for separating fast and slowly changing variables is used here. The terms which do not enable to be integrated without secular terms correspond to critical combinations of multi–indices satisfying the relations

$$\sum_{n=1}^{9}i_n - j_n + k_n - l_n = 1, \qquad \gamma_n = \delta_{9,n} - i_n + j_n - k_n + l_n\,, \qquad (9)$$

$\delta_{i,n}$ being the Kronecker symbol.

They are represented in the similar manner as the series (6) and (7) with (8)

$$U_p^* = \sum{}^* U^p_{i,j,k,l,\gamma}\exp\sqrt{-1}\,(\gamma\lambda)\prod_{n=1}^{9}a_n^{i_n}\bar a_n^{j_n}b_n^{k_n}\bar b_n^{m_n}\,, \qquad (10)$$

$$U_w^* = \sum{}^* U^w_{i,j,k,l,\gamma}\exp\sqrt{-1}\,(\gamma\lambda)\prod_{n=1}^{9}a_n^{i_n}\bar a_n^{j_n}b_n^{k_n}\bar b_n^{m_n} \qquad (11)$$

The asterisk at the summation sign indicates that this summation is taken only over critical values (9). Such terms form an autonomous secular system

$$\dot\alpha = \sqrt{-1}\,N\left[A\alpha + \Phi(\alpha, \bar\alpha, \beta, \bar\beta)\right]\,,$$

$$\dot\beta = \sqrt{-1}\,N\left[B\beta + \Psi(\alpha, \bar\alpha, \beta, \bar\beta)\right] \qquad (12)$$

in slowly changing variables

$$\alpha = (\alpha_1, \ldots, \alpha_9)\;(\alpha_i = a_i\exp-\sqrt{-1}\lambda_i)\,,$$

$$\beta = (\beta_1, \ldots, \beta_9)\;(\beta_i = b_i\exp-\sqrt{-1}\lambda_i)\,. \qquad (13)$$

To complete this system one should add the corresponding conjugate equations.
Here $N = diag(n_1, \ldots, n_9)$, A and B are 9×9 constant matrices of semi-major axes, mean motions and masses of all the bodies under consideration.

9–vectors Φ, Ψ are represented by the power series with quasi–periodic coefficients

$$\Phi = \sum{}^* U^{\Phi}_{i,j,k,l,\gamma}\prod_{n=1}^{9}\alpha_n^{i_n}\bar\alpha_n^{j_n}\beta_n^{k_n}\bar\beta_n^{m_n}\,, \qquad (14)$$

$$\Psi = \sum{}^* U^{\Psi}_{i,j,k,l,\gamma}\prod_{n=1}^{9}\alpha_n^{i_n}\bar\alpha_n^{j_n}\beta_n^{k_n}\bar\beta_n^{m_n}\,, \qquad (15)$$

Functions Φ, Ψ contain only forms of odd degree in slowly changing variables α, $\overline{\alpha}$, β, $\overline{\beta}$ starting with the third degree terms. Since the influence of the Moon on the major planets is not taken into account the system (12) splits into two secular systems for the Moon and for the planets. The last of them is determined in GPT.

5. SOLUTION OF SECULAR SYSTEM

Let μ_j, $S_{.j}$ and ν_j, $T_{.j}$ ($j = 1, 2, \ldots, 9$) be the eigenvalues and eigenvectors of the matrices NA and NB, respectively. The numbers μ_j, ν_j are real and incommensurable but one of ν_j is zero. In virtue of the defining relations

$$NAS_{.j} = \mu_j S_{.j}, \qquad NBT_{.j} = \nu_j T_{.j} \qquad (16)$$

one has

$$S^{-1}NAS = \mu\,, \qquad T^{-1}NBT = \nu \qquad (17)$$

where μ, ν are the diagonal matrices of the corresponding eigenvalues and $S = ||S_{ij}||$, $T = ||T_{ij}||$, S_{ij} , T_{ij} being the components i of the eigenvectors $S_{.j}$, $T_{.j}$, respectively. Hence, the linear transformations

$$\alpha = Sx\,, \qquad \beta = Ty \qquad (18)$$

transform the system (12) into the form

$$\dot x = \mathrm{i}\;[\mu x + NR_1(x, \overline{x}, y, \overline{y})], \qquad \dot{\overline{x}} = -\,\mathrm{i}\;[\mu\overline{x} + NR_2(x, \overline{x}, y, \overline{y})]$$

,

$$\dot y = \mathrm{i}\;[\nu y + NR_3(x, \overline{x}, y, \overline{y})], \qquad \dot{\overline{y}} = -\,\mathrm{i}\;[\nu\overline{y} + NR_4(x, \overline{x}, y, \overline{y})] \qquad (19)$$

with 9–vectors of new variables x , $\overline{x}$, y , $\overline{y}$ and new right–hand members presented by 9–vectors

$$R_1 = N^{-1}S^{-1}N\Phi\,, \qquad R_2 = -\overline{R}_1\,, \qquad R_3 = N^{-1}T^{-1}N\Psi\,, \qquad R_4 = -\overline{R}_3\,. \qquad (20)$$

As indicated above, $\mu = \mathrm{diag}\,(\mu_1, \ldots, \mu_9)$ and $\nu = \mathrm{diag}\,(\nu_1, \ldots, \nu_9)$ are the diagonal matrices of the proper frequencies of the motion of the perihelia and nodes, respectively. Their values are given in the table 1.

Table 1. Secular planetary–lunar motions

j	$-\mu_j$("/y)	ν_j("/y)
1	5.4616712	5.2010621
2	7.3466191	6.5706750
3	17.331990	18.746646
4	18.005604	17.637376
5	3.7302930	0
6	22.484245	25.956797
7	2.7135503	2.9168219
8	0.6356290	0.6802239
9	140942.45	69509.346

For the planets they were put into conformity with (Bretagnon, 1974). In accordance with this paper the number of the zero secular eigen value is $N = 5$. Components μ_9 and ν_9 relating to the Moon are much greater by their absolute values than components μ_j, ν_j ($j = 1, \ldots, 8$) relating to the planets.

We suppose that the solution of the planetary system is known from GPT in the form

$$\alpha_i = \sum_{j=1}^{8}S_{ij}\eta_j^{(1)}\exp\sqrt{-1}\omega_j^{(1)}\,,$$

$$\beta_i = \sum_{j=1}^{8}T_{ij}\eta_j^{(2)}\exp\sqrt{-1}\omega_j^{(2)}\,. \qquad (21)$$

$\omega_j^{(1)}) = \mu_j\,t + \chi_j^{(1)}$, $\omega_j^{(2)}) = \nu_j\,t + \chi_j^{(2)}$,
$\eta_j^{(1)}, \eta_j^{(2)}, \chi_j^{(1)}, \chi_j^{(2)}$ are the arbitrary integration constants. In this paper they are taken from (Bretagnon, 1974) and $\eta_j^{(1)}, \eta_j^{(2)}$ are shown in the table 2 together with corresponding values for the Moon.

Table 2. Integration constants

j	$\eta_j^{(1)}$	$\eta_j^{(2)}$
1	0.18141040	0.06274851
2	0.01909712	0.00506380
3	0.01056860	0.01222166
4	0.07300403	0.02519918
5	0.04319426	0.01383974
6	0.04837743	0.00786789
7	0.03140786	0.00880286
8	0.00923780	0.00588386
9	0.05490049	0.04484172

The resulting secular system system (19) is solved by Birkhoff normalization technique. One may rewrite this system in the form of a single equation

$$\dot X = \mathrm{i}\big[PX + NR(X)\big]. \qquad (22)$$

$X = (x, \overline{x}, y, \overline{y})$ has 36 components.
P is diagonal 9×9 matrix of the structure

$$P = diag(\mu\,, -\mu\,, \nu\,, -\nu), \qquad (23)$$

N being again 9×9 diagonal matrix of the mean motions n_i.
The transformation from X to new variables $Y(u, \overline{u}, v, \overline{v})$ (u, v are 9–vectors)

$$X = Y + \Gamma(Y) \qquad (24)$$

results in a new system

$$\dot Y = \mathrm{i}\big[PY + NF(Y)\big]. \qquad (25)$$

Functions Γ and F representing the series in powers of Y are found by iterations from the equations

$$U = R - N^{-1}\Gamma_Y NU^*\,, \qquad (26)$$

$$\Gamma_Y PY - P\Gamma = NU^+\,, \qquad (27)$$

$$U = U^* + U^+\,, \qquad F = U^*\,. \qquad (28)$$

The Jacobi matrix Γ_Y composes of the derivatives of the components of Γ with respect to the variables Y. The splitting of U is aimed to ensure the integration of (27) without t–secular terms.
Thus, Birkhoff normalization consists of constructing the formal power series (24) reducing (22) to the system (25) with the power series F admitting the straightforward integration of this system.

The secular system contains only forms of odd degree and each step of the iteration process increases the accuracy of the determination of U and Γ by two orders. The iterations start with $U = R$. Therefore, U may be represented by power series in the form

$$U_\kappa = \sum U^{(\kappa)}_{i,j,k,l}\prod_{j=1}^{9}u_j^{i_j}\overline{u}_j^{k_j}v_j^{l_j}\overline{v}_j^{m_j}\,, \qquad \kappa = 1, 2 \qquad (29)$$

under first condition (9) on the multi–indices.
U^*_κ ($\kappa = 1, 2$) includes again all the terms of U_κ (29) which do not enable to be integrated in the trigonometric form. This case corresponds to critical relations of the multi–indices

$$i_j = k_j + \delta_{9,j}\delta_{\kappa,1}, \qquad l_j = m_j + \delta_{9,j}\delta_{\kappa,2} \qquad (30)$$

Then U^*_κ is represented in the form

$$U^*_\kappa = s_\kappa\sum{}^* U^{(\kappa)}_{i,j,k,l}\prod_{j=1}^{9}(u_j\overline{u}_j)^{k_j}(v_j\overline{v}_j)^{m_j} \qquad (31)$$

where $s_1 = u$, $s_2 = v$, $\sum^*$ means that the summation is performed over critical multi–indices satisfying the relations (30).
Γ_κ is found from the equation (27) with taking into account (29) and (31) in the similar form as (29) with coefficients

$$\Gamma^{(\kappa)}_{i,j,k,l} = \frac{nU^{+(\kappa)}_{i,j,k,l}}{\sum_{j=1}^{9}\left[(i_j - k_j - \delta_{9,j}\delta_{\kappa,1})\,\mu_j + (l_j - m_j - \delta_{9,j}\delta_{\kappa,2})\,\nu_j\right]}\,. \qquad (32)$$

All the denominators are not equal to zero because of the incommensurable eigenvalues and the absence of the terms with critical characteristics (30). Therefore, Γ_1 and Γ_2 are the known values. As a result, the equations (25) with using (28) and (31) are transformed into the system of linear equations

$$\dot u_9 = \sqrt{-1}\,u_9\,(\mu_9 + \delta\mu_9), \qquad \dot v_9 = \sqrt{-1}\,v_9\,(\nu_9 + \delta\nu_9)\,. \qquad (33)$$

$\delta\mu_9$ and $\delta\nu_9$ are real constant corrections to corresponding eigenvalues for the Moon.
Hence, these equations admit the straightforward integration and the solution has the form

$$u = \eta_{1,9}\,\exp\sqrt{-1}\,\varphi_{1,9}, \qquad v = \eta_{2,9}\,\exp\sqrt{-1}\,\varphi_{2,9}, \qquad (34)$$

$$\begin{pmatrix}\varphi_{1,9}\\ \varphi_{2,9}\end{pmatrix} = \begin{pmatrix}\mu_9 + \delta\mu_9\\ \nu_9 + \delta\nu_9\end{pmatrix}t + \begin{pmatrix}\tau_{1,9}\\ \tau_{2,9}\end{pmatrix}, \qquad \begin{pmatrix}\delta\mu_9\\ \delta\nu_9\end{pmatrix} = n\sum{}^*\begin{pmatrix}U^1_{km}\\ U^2_{km}\end{pmatrix}\prod_{i=1}^{9}\eta_{1,i}^{2k_i}\eta_{2,i}^{2m_i} \qquad (35)$$

with real constants of integration $\eta_{j,i}$ and $\tau_{j,i}$ ($j = 1, 2\,;\, i = 1, \ldots, 9$). $\eta_{j,i}$ are shown in the table 2. For the Moon $\eta_{1,9} = e$, $\eta_{2,9} = \sin\frac{I}{2}$ where e and I is the eccentricity and inclination of the lunar orbit, respectively.
The resulting solution of the secular system for the Moon has the form

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \sum\begin{pmatrix}\alpha_{klmn}\\ \beta_{klmn}\end{pmatrix}\prod_{i=1}^{9}\exp\sqrt{-1}\Big[(k_i - l_i)\,\varphi_{1,i} + (m_i - n_i)\,\varphi_{2,i}\Big]\,, \qquad (36)$$

$$\sum_{i=1}^{9}(k_i - l_i + m_i - n_i) = 1.$$

The trigonometric solution of the secular system has the semi–analytical form with numerical coefficients α_{klmn} and β_{klmn}. It includes terms due to the secular evolution of the lunar perigee and node as well as of that of the major planets.

6. REFERENCES

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