A generalized theory of the figure of the Earth: application to the moment of inertia and global dynamical flattening

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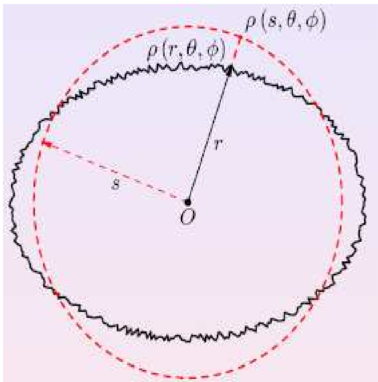

JSR, Paris, 16-18/09/2013

Outline

- **Motivations**
- **new theory of the figure of the earth:**

  a generalization
- **Applications:**
  - **the interior equilibrium figures- Geoid / CMB;**
  - **MoI (A/B/C) & H**

Motivations

- **The difference of the global dynamical flattening (H) between the theoretical values & observations**

- Need the non-symmetric figures of equi-density layers (esp. geoid, CMB) throughout the earth for global geophysical study, e.g., nutation / FCN - pressure torque@CMB (V.D)

- the problems in the traditional theory of the figure of the earth

Status & problems (1):
problems in the global dynamic flattening (H)

$$H = \frac{C - \frac{A+B}{2}}{C}$$

$$C = \int_v (x^2 + y^2)\, \rho\, \mathrm{d}V$$

H is related to the precession, main nutation, tilt-over-mode, …

The traditional theory of the figure of the earth + PREM

➔ H≈1/308.8

precise precession obs.

➔ H≈1/305.5

~1.1% difference

The report of IAU WG ”nutation”, 2000;
《The solved & unsolved problems in Earth rotation》
Refs.: Defraigne, Dehant, Capitaine, …

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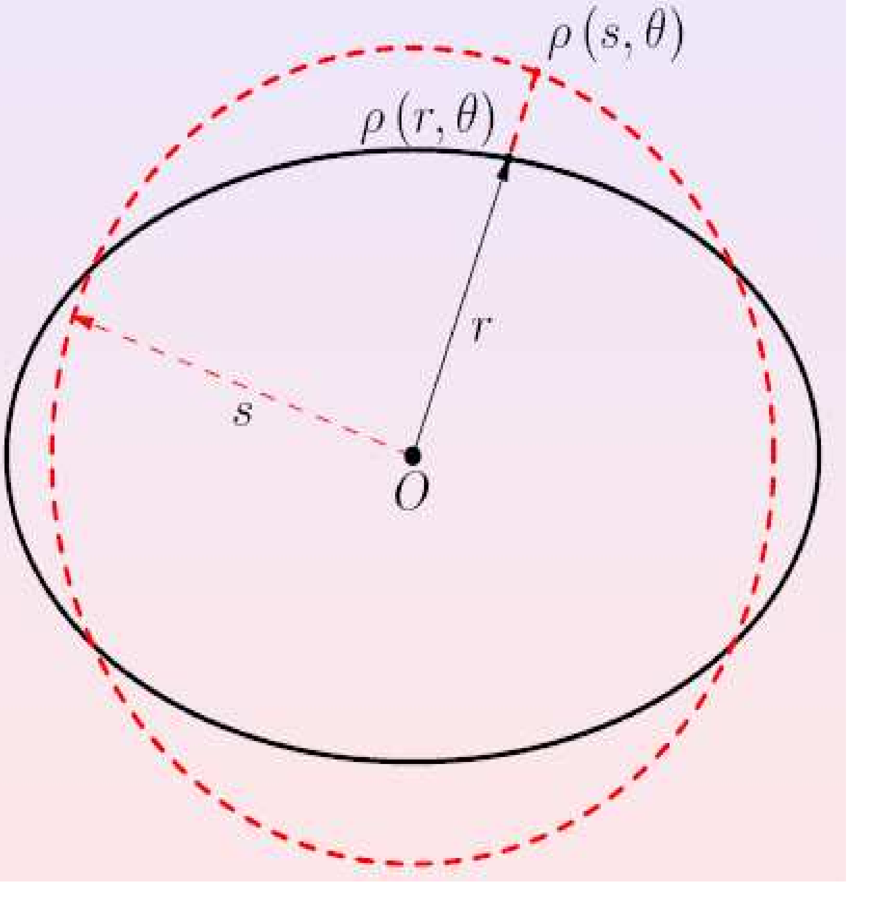
- **the problems in the traditional theory of the figure of the earth**

Status & problems (2):
problems in the traditional theory of the figure of the earth

Clairaut (1743) theory (1st order theory)

$$r(s,\theta,\phi) = s[1 - \frac{2}{3} f P_2(\cos\theta)]$$

$$\frac{d^2 f}{dq^2} + \frac{6}{q}\frac{\rho}{D}\frac{df}{dq} - \frac{6}{q^2}\left(1 - \frac{\rho}{D}\right) f = 0$$



Status & problems (2)

- **Darwin(1899) - de Sitter(1924) theory (2^{nd} Order)**

$$r = s\left[1 - f\cos\theta - \left(\frac{3}{8}f^2 + k\right)\sin^2 2\theta\right]$$

$$\beta^2\ddot{\kappa} + 6\frac{\delta}{D}\beta\dot{\kappa} + \left(-20 + 6\frac{\delta}{D}\right)\kappa = 3\left(1 - \frac{\delta}{D}\right)e^2 + \\ + \left(1 - \frac{9}{2}\frac{\delta}{D}\right)\beta e\dot{e} - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)\beta^2\dot{e}^2 \quad .$$

Ref: Moritz H., 《The figure of the Earth》

Status & problems (2)

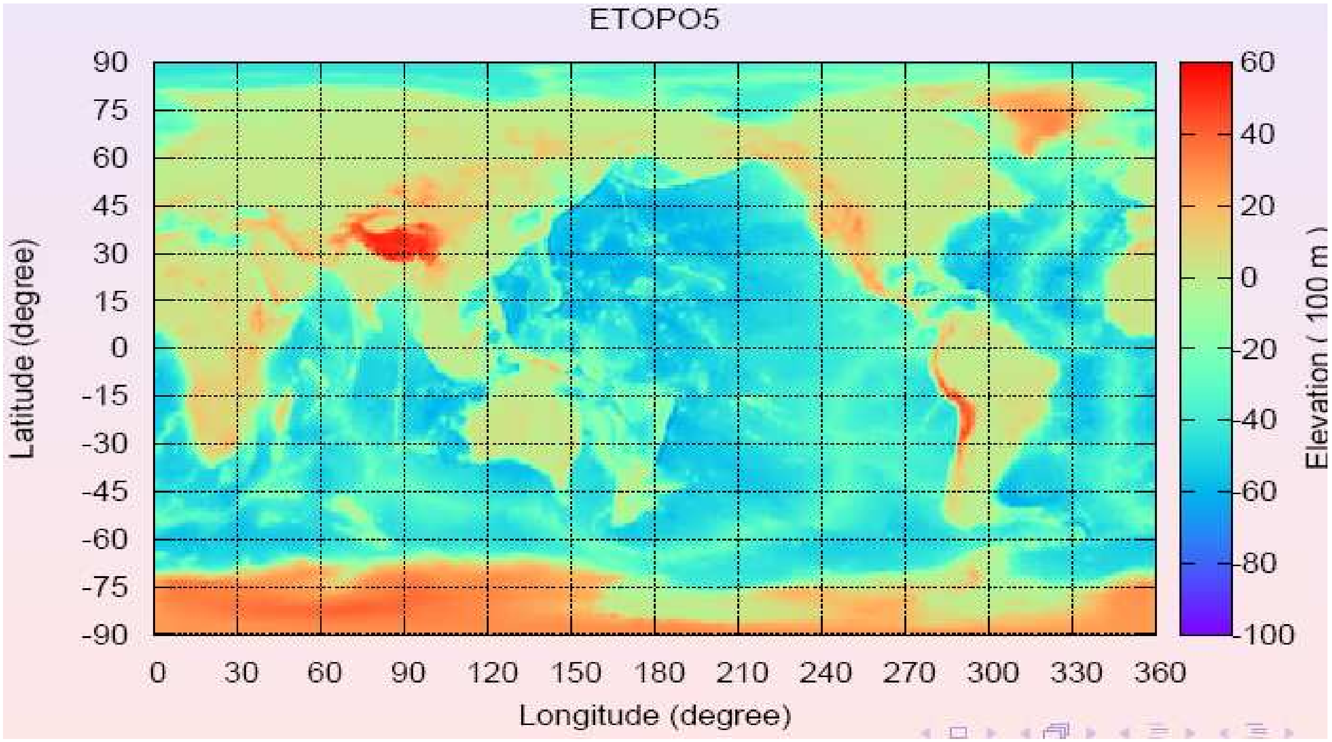
- **Denis (1989) theory (3rd order theory)**

$$r = s\left[1 + s_2 P_2 + s_4 P_4 + s_6 P_6\right]$$

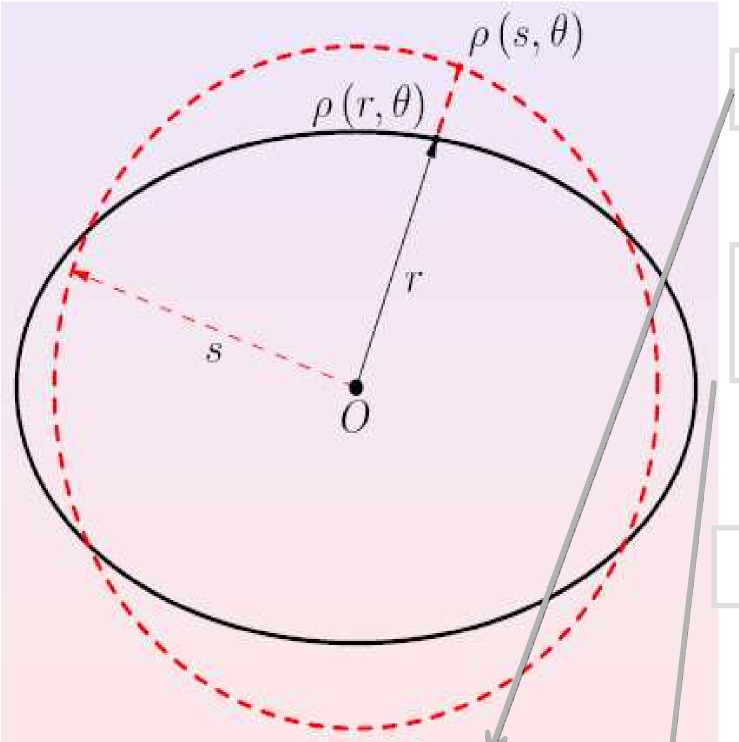
Status & problems (2)

All assume that the earth is of rotating symmetric (m=0) & equatorial symmetric (n=even).

But our real earth is obviously NOT! but of topography



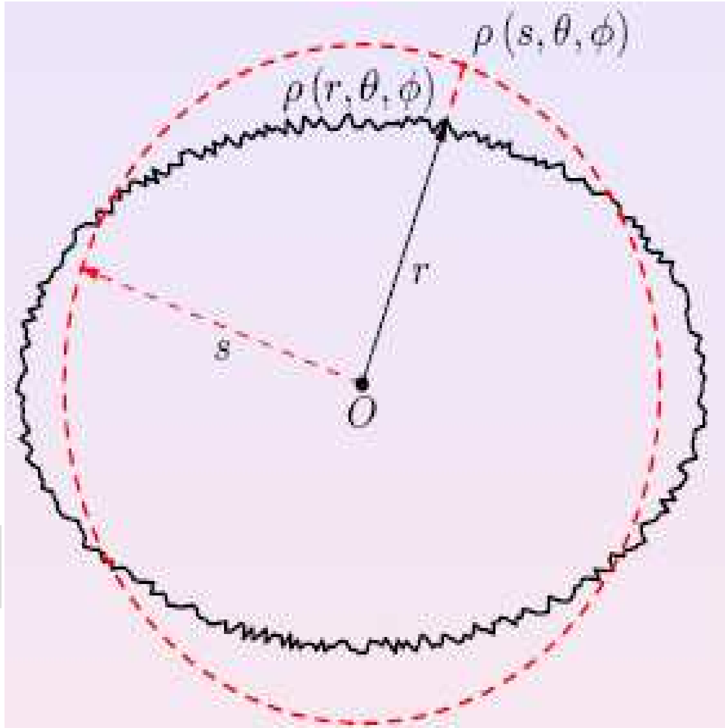
A generalized theory of figure of the earth



Clairaut,1743

Darwin,1899;
De Sitter,1924

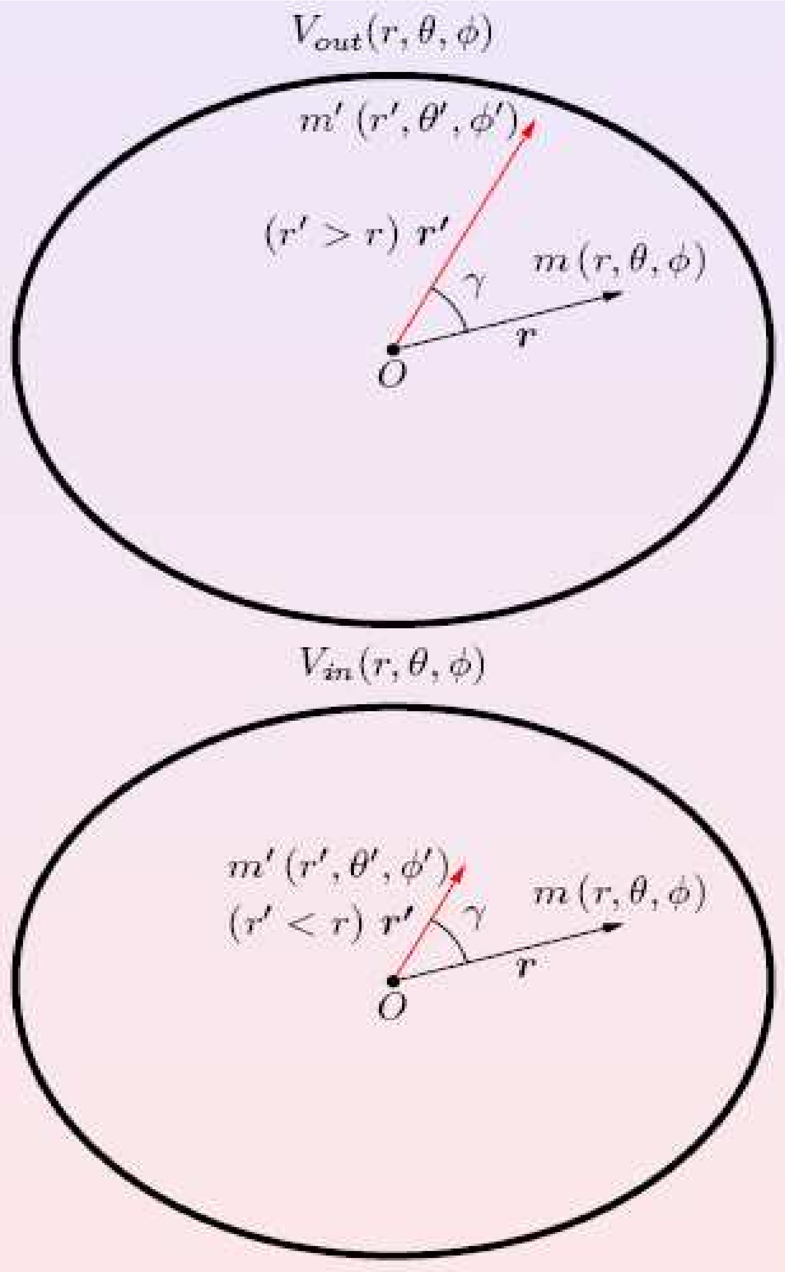
Denis,1985-2006



$$r = s\left[1 + s_2 P_2 + s_4 P_4 + s_6 P_6\right]$$

$$r(s,\theta,\phi) = s\left[1 + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} H_n^m(s)\, Y_n^m(\theta,\phi)\right]$$

Calculation of gravity potential

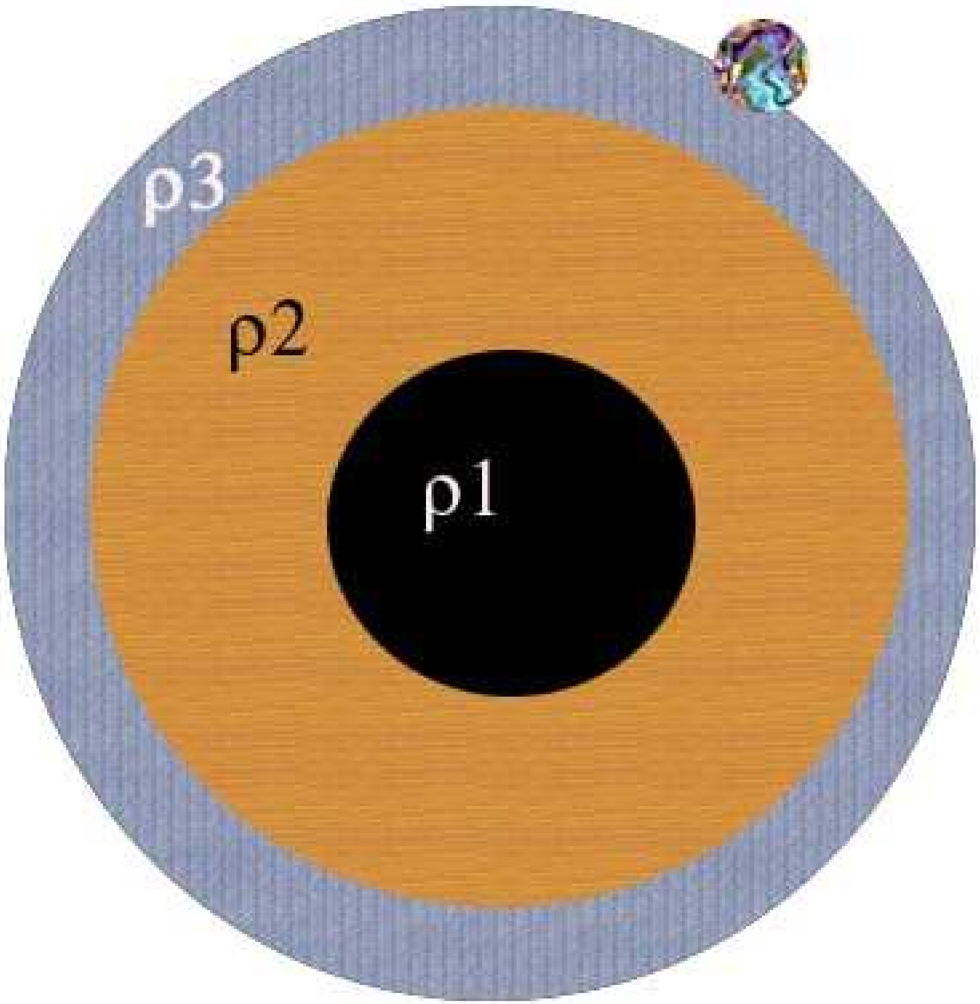


$$W(r,\theta,\phi) = V_{in}(r,\theta,\phi) + V_{out}(r,\theta,\phi) + Z(r,\theta,\phi)$$

V_{in} / V_{out} : gravitational potential by the mass inside / outside the target equi-potential surface

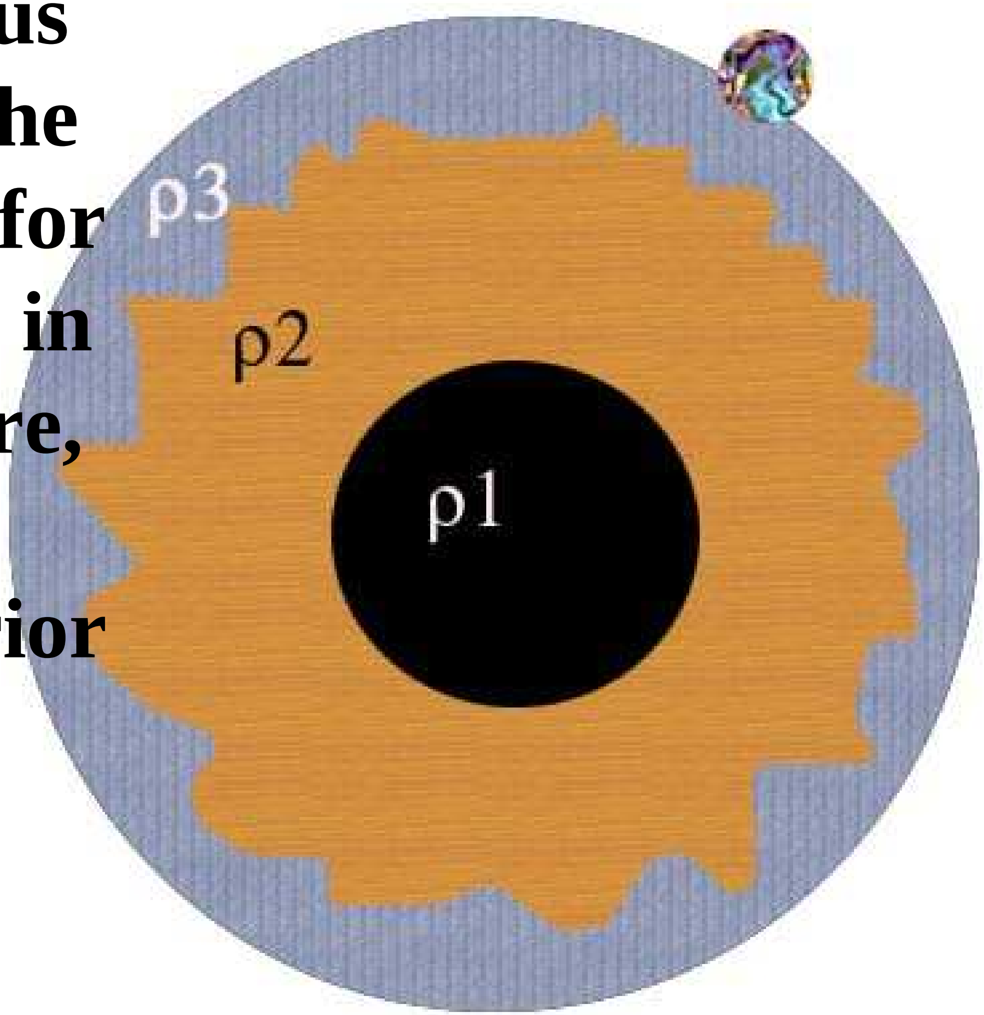
Z: centrifugal potential

Direct + indirect effects



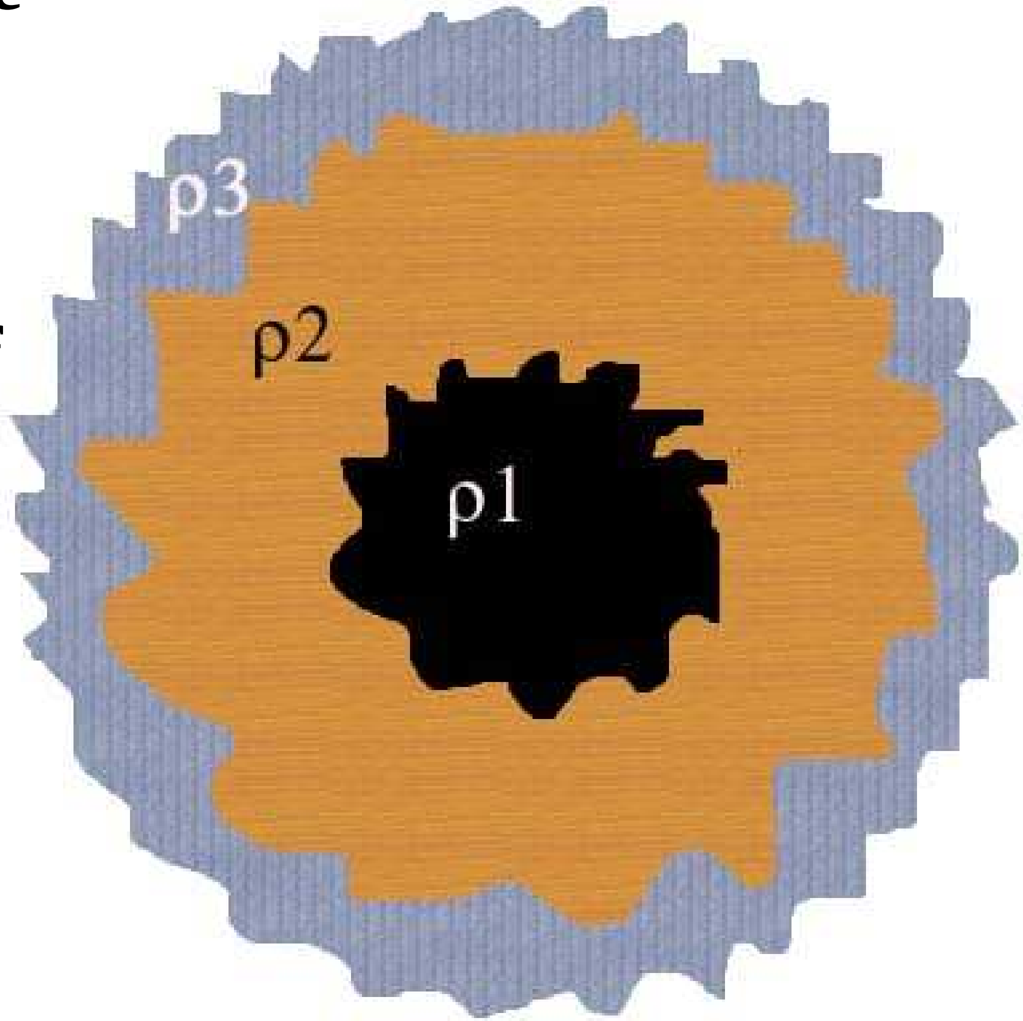
Direct effect

the crust inhomogeneous mass change directly the gravitational potential for all mass points interior in different ways, therefore, the figure of equi-potential surfaces interior are changed without symmetries.



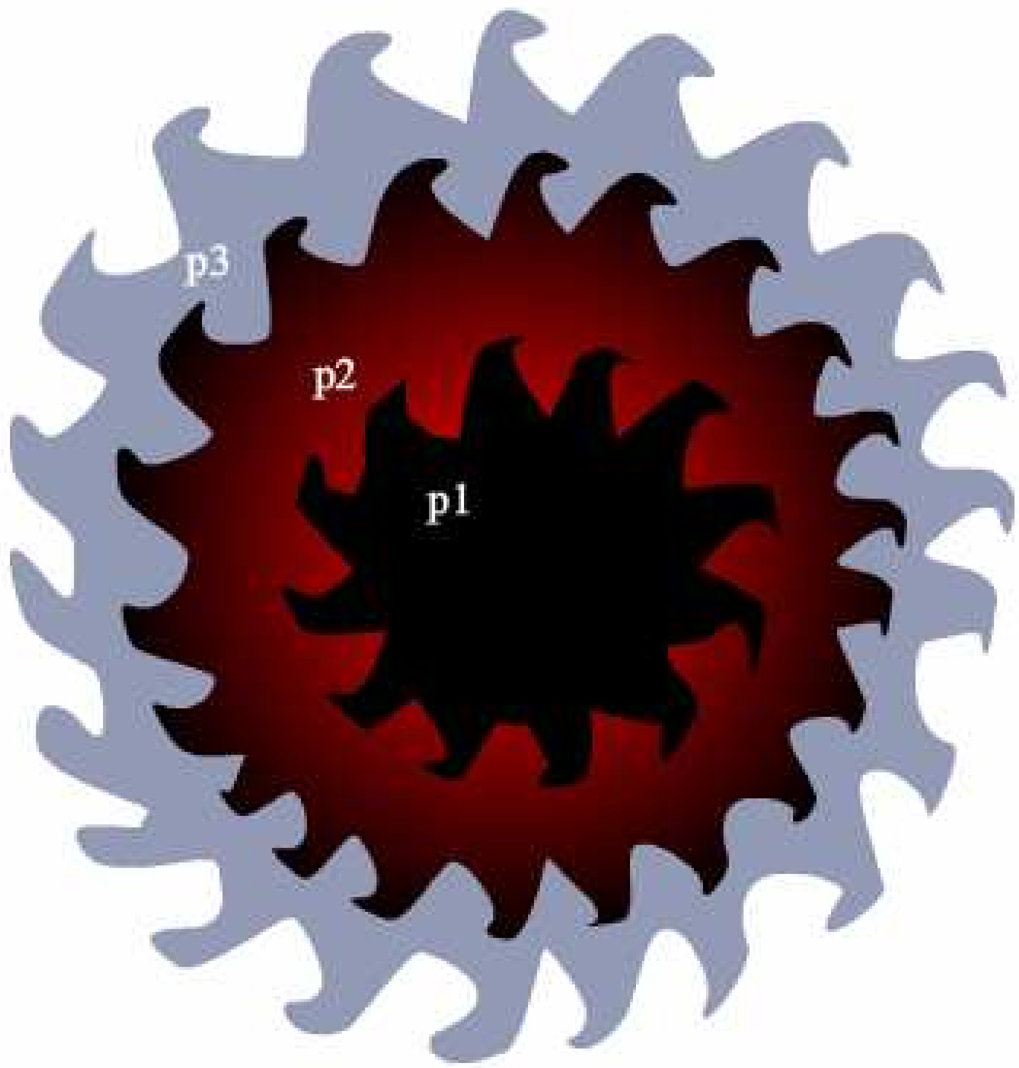
Indirect effect

As the figures of equi-density surfaces (then the density distribution) interior are changed by the direct effect, the gravitational potential of other locations (outside/inside this surface) are changed, and the figures of equi-potential surfaces all through the earth are then changed again.



Indirect effect

This process is reciprocal and needs iteration, and will finally reach equilibrium.



The final eqs. of the figure of the equi-potential surfaces

$$
\begin{aligned}
W &= V_{in} + V_{out} + Z \\
&= GE_0(s) + G\bar{\rho}s^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m \left[m_h p_{n,m} + \sum_{l=0}^{\infty} \frac{s^{l-2}}{\bar{\rho}} u_{l,n,m} \right. \\
&\qquad \left. + \sum_{l=1}^{\infty} g_{l,n,m} + \sum_{l=0}^{\infty} f_{l,n,m} \right] \\
&= GE_0(s) + G\bar{\rho}s^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\theta, \phi)\Xi_n^m(s) \\
\Xi_n^m &= m_h p_{n,m} + \sum_{l=0}^{\infty} \frac{s^{l-2}}{\bar{\rho}} u_{l,n,m} + \sum_{l=1}^{\infty} g_{l,n,m} + \sum_{l=0}^{\infty} f_{l,n,m}
\end{aligned}
$$

$$
\begin{cases}
\Xi_n^m + (-1)^m \Xi_n^{-m*} = 0 \\
n = 1, \ldots, \infty \\
m = 0, \ldots, n
\end{cases}
$$

Eqs./parameters: $(n+1)^2\times$ No._layers

Strategy to solve the eqs.

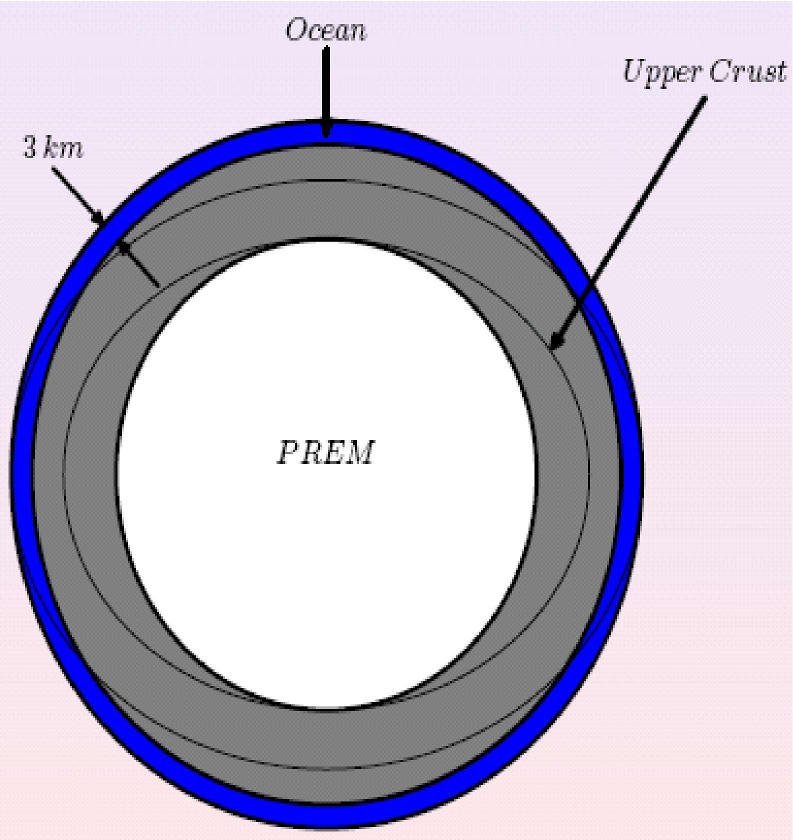
•**Truncated to n & m=6: 3-rd order precision**

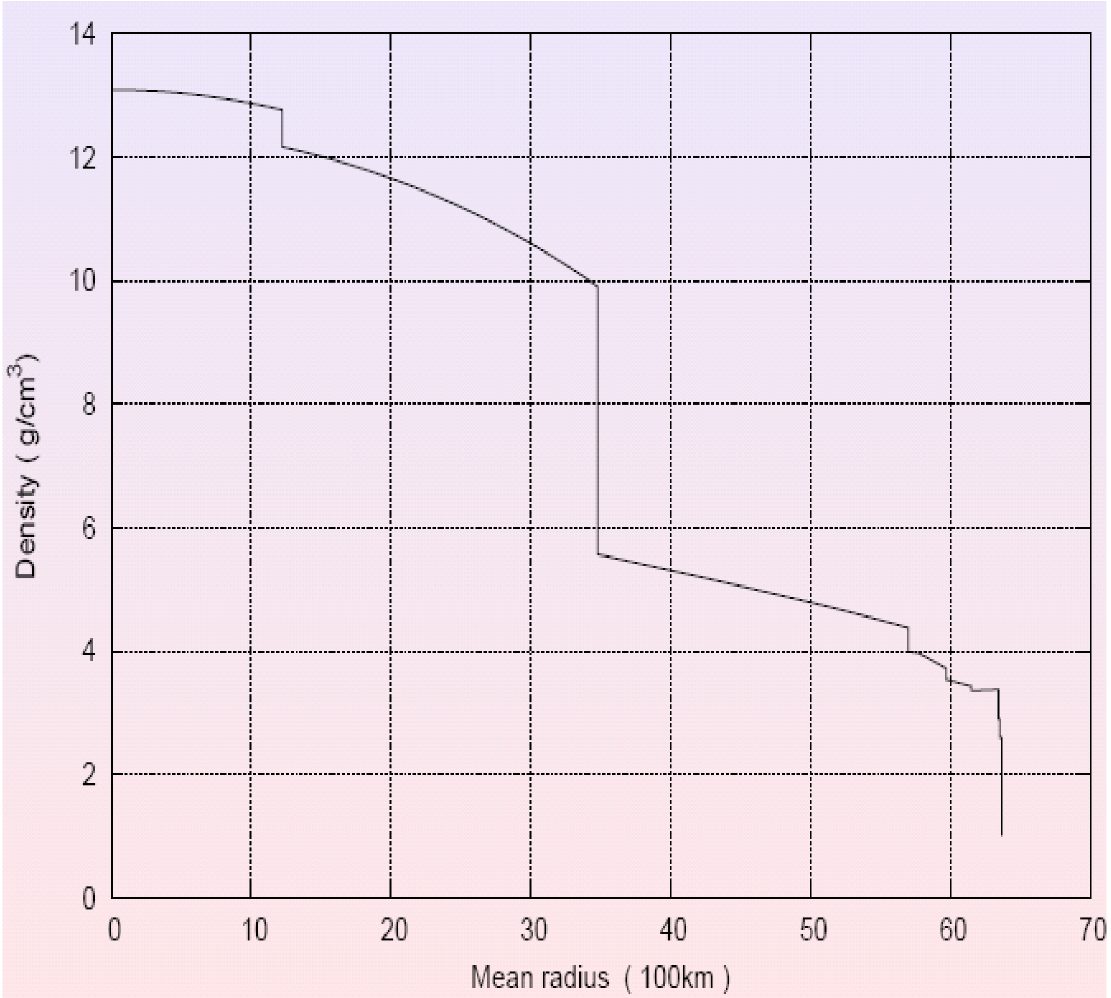
truncated to 8: no obvious change at 3rd. order precision?

•**Detail derivation for the formulas of the hundrads parameters are done by symbol processing software "Mathmatica"**

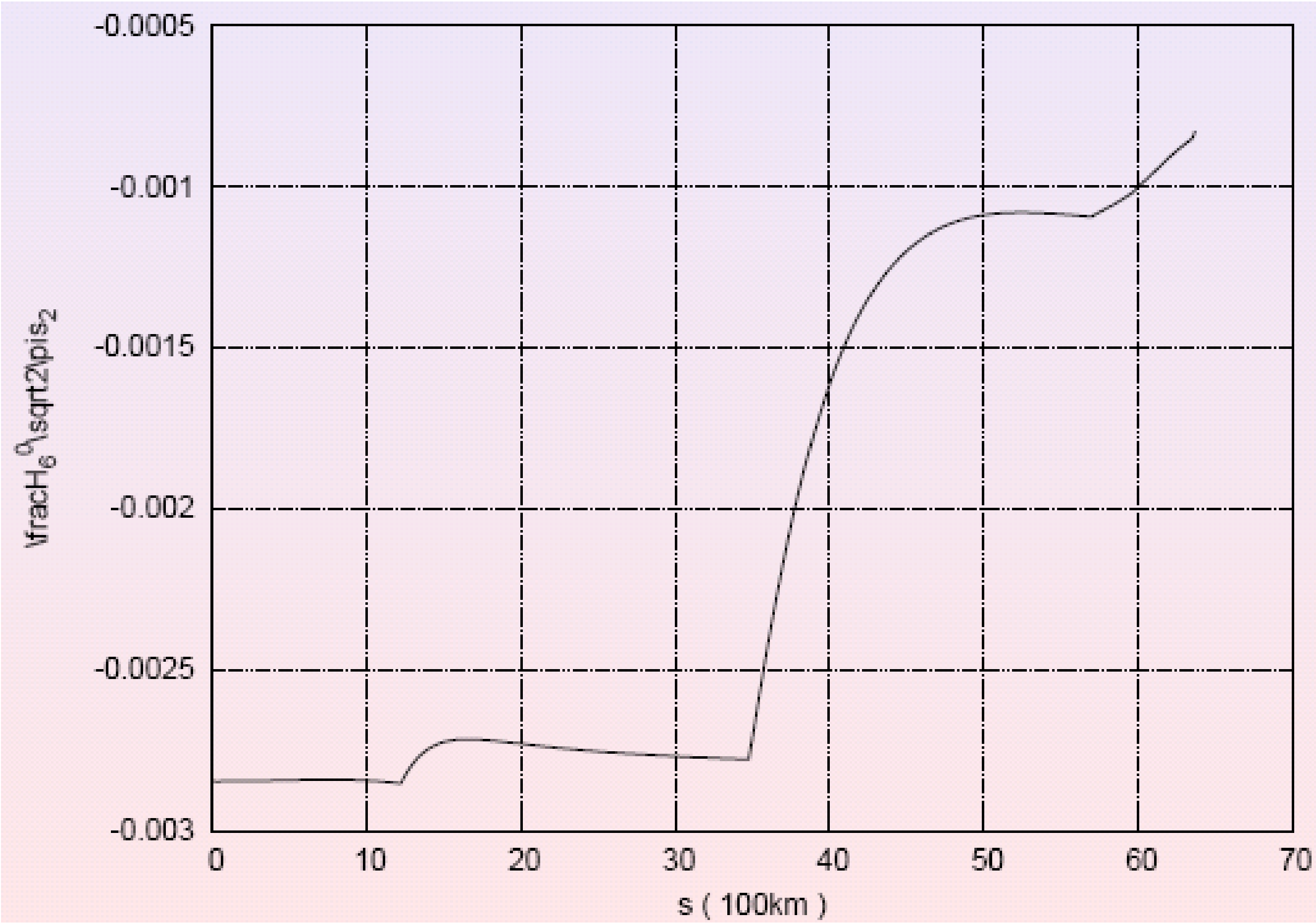
•**the parameters for real earth are then computed from the center to the outer surface in the self-consistent formulas by *iterations*.**

Earth model: PREM (Dziewonski & Anderson, 1981)





Validation of this new theory: degenerate to symmetric earth (PREM)



Agree with Denis(1989) very well.

The relative diff. between H_6^0 & S_6

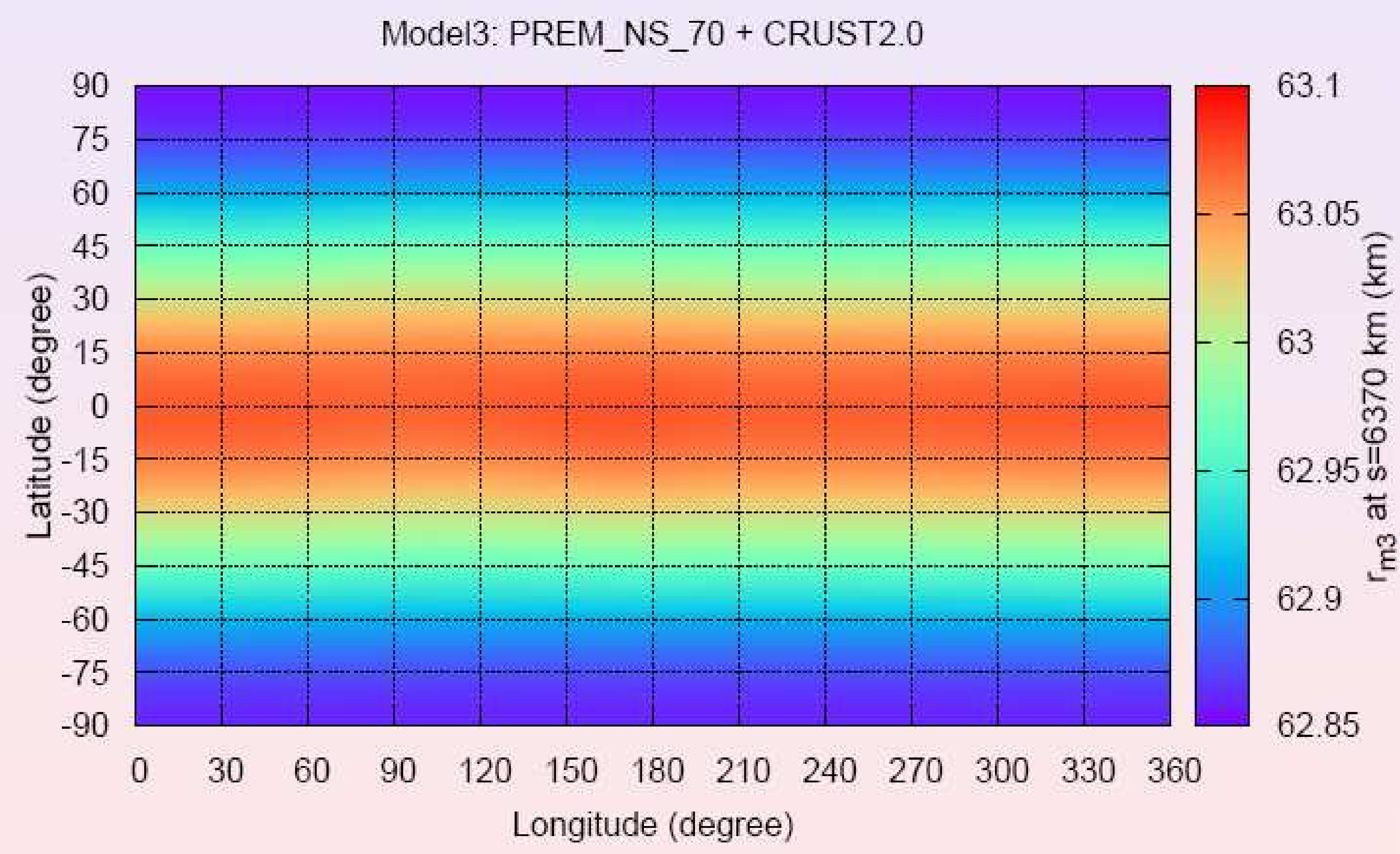
$$\left(H_6^0(s)\sqrt{\frac{13}{4\pi}} - s_6(s) \right) / s_6(s)$$

Models for the more real Earth

the ocean/topography models used here

	ECCO	GTOPO30	ETOPO5	CRUST2.0
Source	NOPP	USGS	NOAA	Chulick etc.
Layers no.	46	–	–	8
Depth(km)	5.615	0	10.376	70.137
Grid res.	不均匀	$30'' \times 30''$	$5' \times 5'$	$2^\circ \times 2^\circ$

Results: the profiles of the interior ‘geoid’



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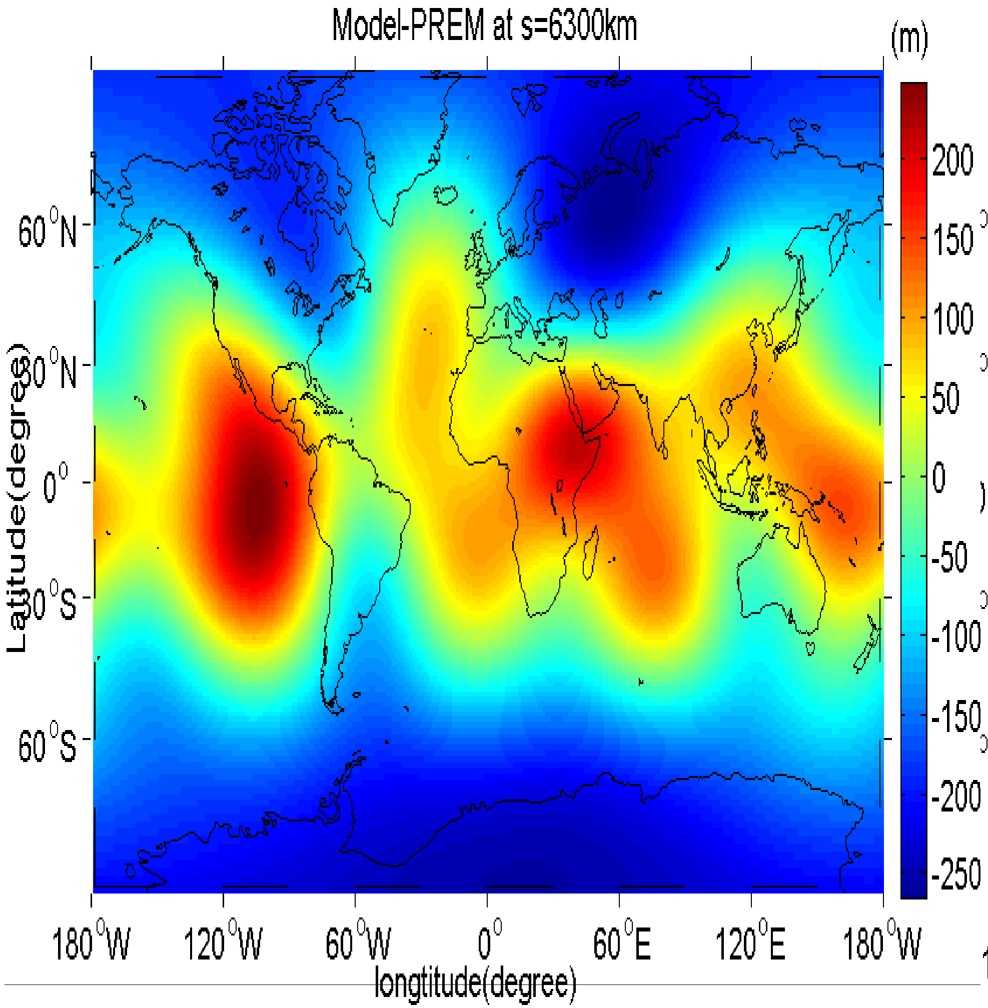

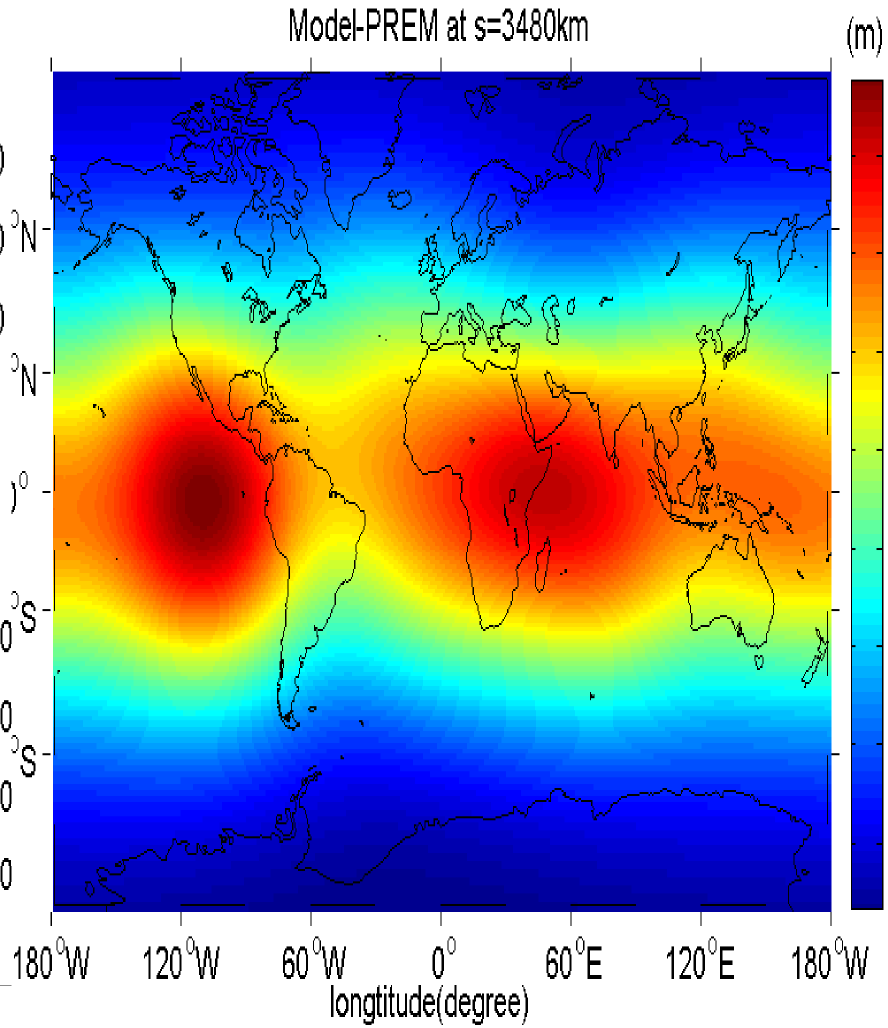

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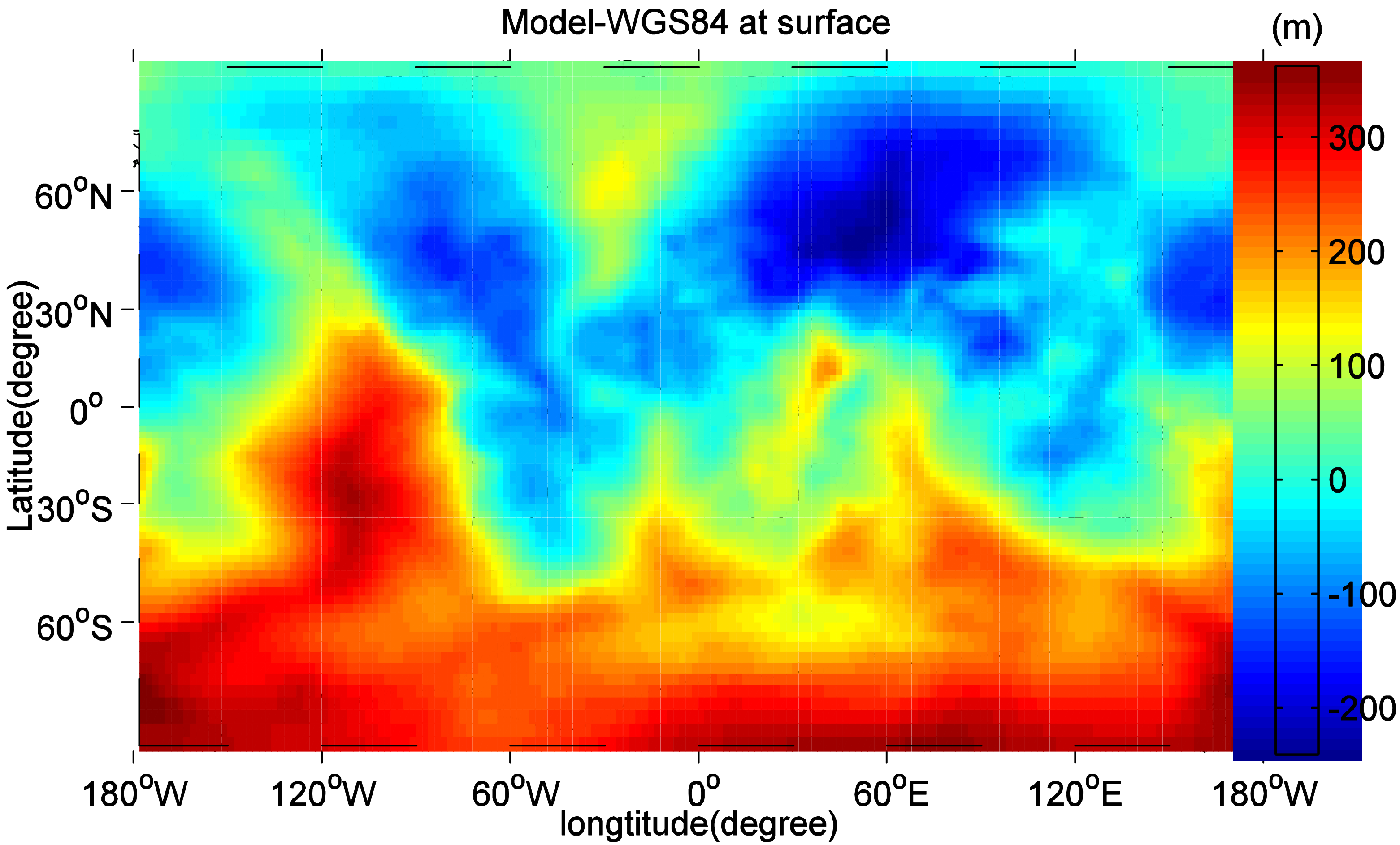

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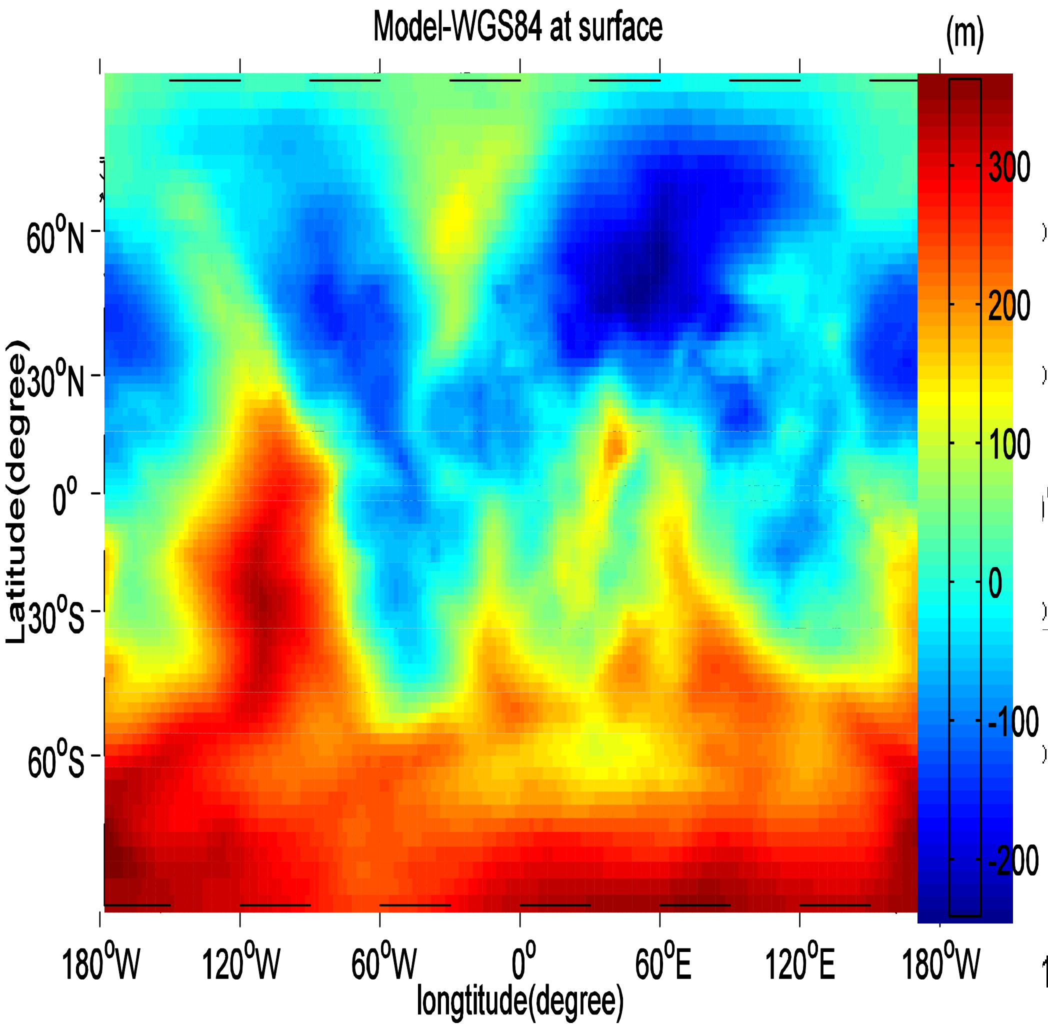
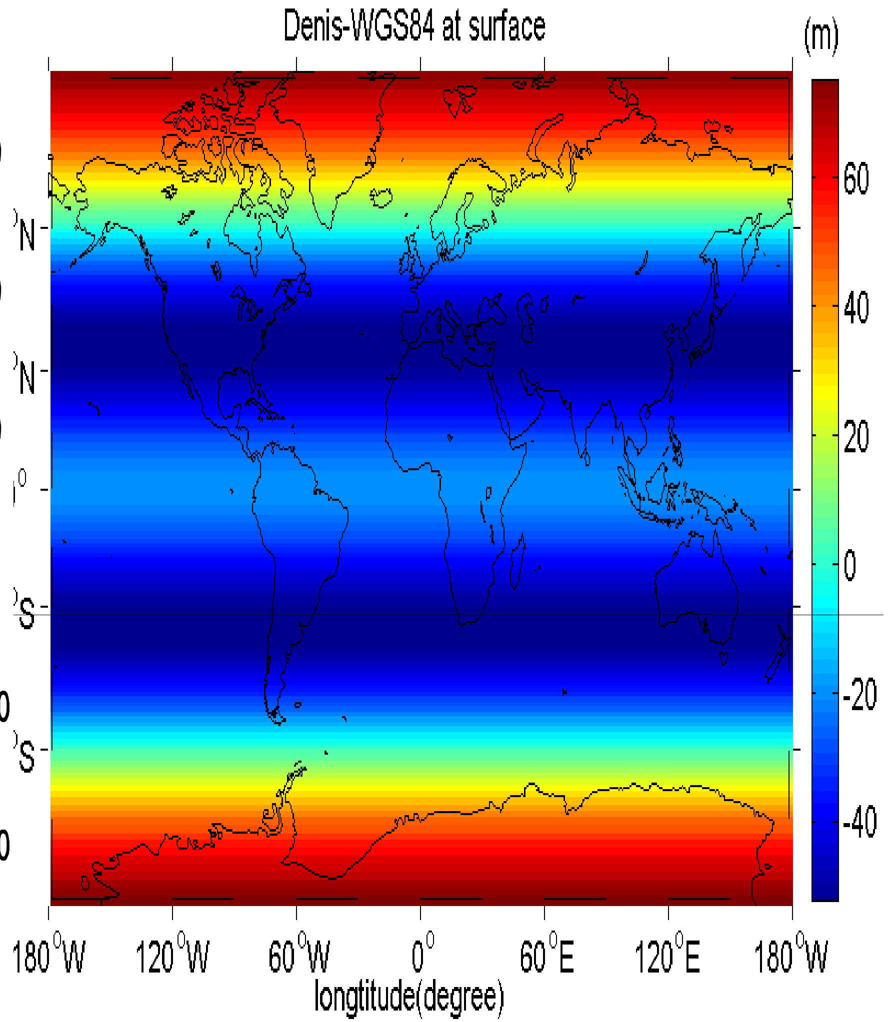
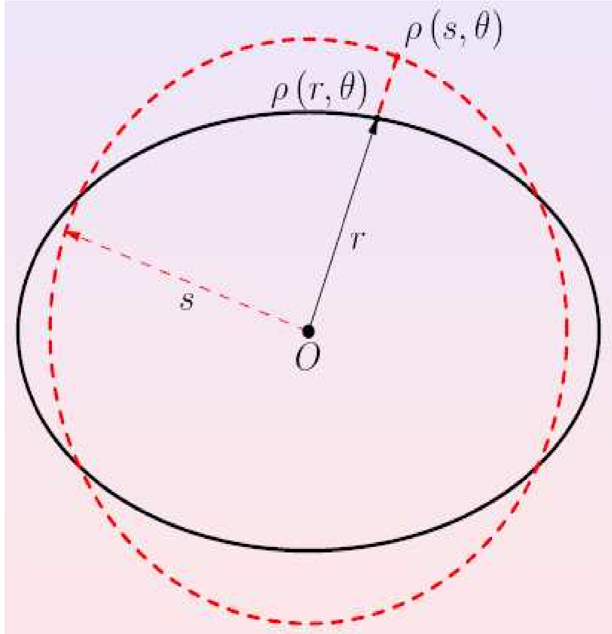




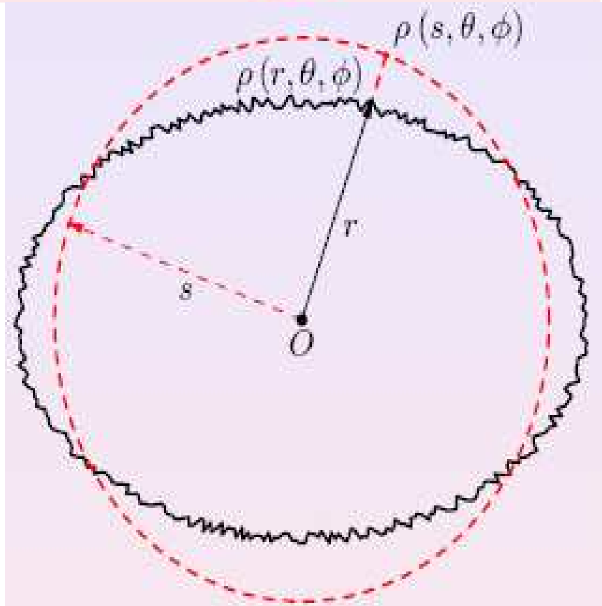
Results: H

$$H = \frac{C - \frac{A+B}{2}}{C}$$

$$C = \int_v (x^2 + y^2)\, \rho\, \mathrm{d}V$$



- Direct contribution of the topo. to H: change the mass distribution in the integral (in the top layers only)



- Indirect contribution of the topo. to H: change the figures of equi-density surfaces interior, then change the density distribution (all through the earth!)

Results: MoI & H

Obs.: 305.5

- Direct effect considered only

	A $10^{37}\ kg \cdot m^2$	B $10^{37}\ kg \cdot m^2$	C $10^{37}\ kg \cdot m^2$	1/H
PREM(-71km)	7.7087284	7.7087284	7.7336553	
CRUST2.0	0.2949340	0.2947971	0.2956929	
TOTAL	8.0036624	8.0035255	8.0293482	311.7674842

- Both direct & indirect effects

	A(10^{37}kg m^2)	B(10^{37}kg m^2)	C(10^{37}kg m^2)	1/H
PREM	**8.0115651**	**8.0115651**	**8.0376170**	**308.52**
This work	**8.0112300**	**8.0114003**	**8.0375719**	**306.12**
EGM2008[17]	**8.0100829**	**8.0102594**	**8.0364807**	**305.46**

+1%

+0.2%

Summary & remarks

1. **A new generalized integrated formula to obtain the equilibrium figures to 3rd-order accuracy for real earth is developed. All the non-zero order and odd degree terms are included in the SH expression of the figures.**

2. **In these formulas, both the direct & indir. contributions of the anti-symmetric crust layers are included.**

3. **Profiles of the equilibrium figures, no longer symmetric, interior the real Earth are obtained; and comparison among them provides an indirect evidence & support for the theory of isostasy**

Summary & remarks

4. The calculated geoid embodies stronger topographical signal than that calculated by traditional theory.

5. The direct effect of the real ocean and topo. layers up to 71km depth changes H by ~0.7% in opposite direction; while the indirect effect, based on this work, can draw back the difference of H_{theory}-$H_{obs.}$ from 1.1% to 0.2%.

Excited? !

Thank You!

Tab. The MoI & H of different models

	A(10^{37}kg m^2)	B(10^{37}kg m^2)	C(10^{37}kg m^2)	1/H
PREM	8.0115651	8.0115651	8.0376170	308.52
PREM-71KM + CRUST2.0 (*)	8.0112300	8.0114003	8.0375719	306.12
PREM-80KM + CRUST2.0	8.0149750	8.0149978	8.0413399	305.13
EGM2008[17]	8.0100829	8.0102594	8.0364807	305.46

I_{yx}

Tab. The MoI of the Earth calculated by two grid integration methods

	A(10^{37}kg m^2)	B(10^{37}kg m^2)	C(10^{37}kg m^2)
PREM-71	7.7164775	7.7164823	7.7418221
Method①	0.2948928	0.2947797	0.2957280
MoI	8.0113703	8.0112620	8.0375501
Method②	0.2948918	0.2947790	0.2957497
MoI	8.0113693	8.0112612	8.0375718