The Time Transfer Function as a tool to compute Range, Doppler and astrometric observables

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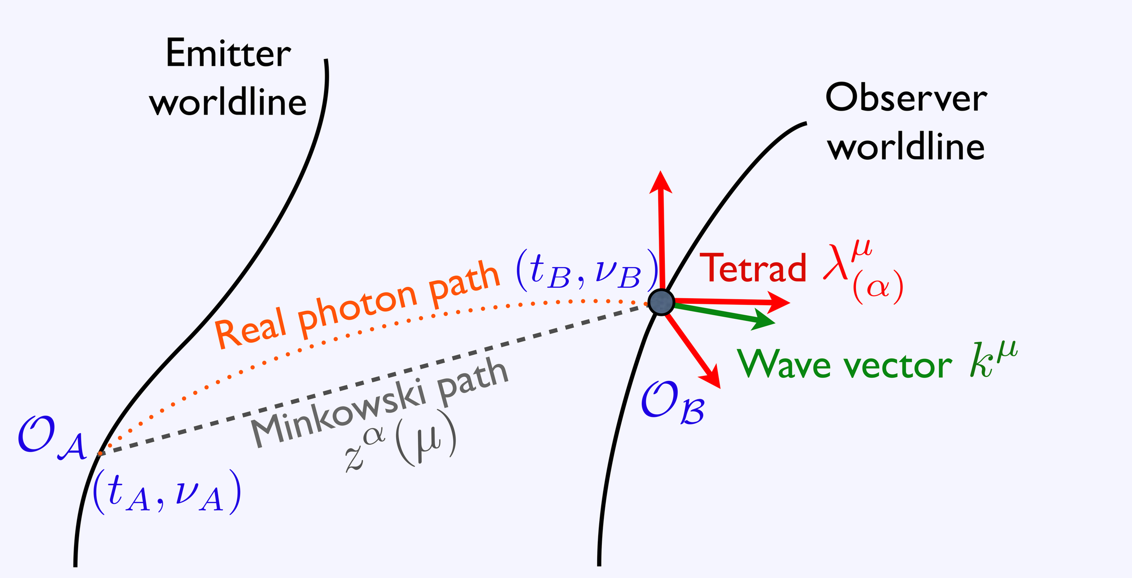
ABSTRACT

In this poster, we will show how the Time Transfer Function (TTF) can be used in the relativistic modeling of range, Doppler and astrometric observables. We will present a method to compute these observables up to second Post-Minkowskian order directly from the space-time metric $g_{\mu\nu}$ without explicitly solving the null geodesic. The resulting expressions involve integrals of some functions defined by the metric over a straight line between the emitter and the receiver of the electromagnetic signal. Some examples will be given within the context of future space missions.

I. Model

Let us consider two observers $\mathcal{O}_{\mathcal{A}/\mathcal{B}}$ moving along their respective worldlines. The first observer sends an electromagnetic signal to the second one. The signal is emitted at the coordinates $(t_A, \mathbf{x_A})$ and has a frequency ν_A. It is received by $\mathcal{O}_{\mathcal{B}}$ at the coordinates $(t_B, \mathbf{x_B})$, with a frequency ν_B. The incident direction of the received signal with respect to a comoving tetrad $\lambda^\mu_{(\alpha)}$ is denoted by $n^{(i)}$.



II. Relation between observables and the Time Transfer Function

A. Time Transfer

The coordinate travel time of a light ray connecting the emission and the reception points is given by the Time Transfer Function $\mathcal{T}_r$ [1, 2]:

$$t_B - t_A = \mathcal{T}_r(\mathbf{x}_A(t_A), t_B, \mathbf{x}_B). \tag{1}$$

This implicit equation can be solved iteratively in the case of a moving emitter.

B. Frequency shift

It can be shown that the expression for the frequency shift can be written as [3, 4]

$$\frac{\nu_B}{\nu_A} = \frac{\left[g_{00} + 2g_{0i}\,\beta^i + g_{ij}\,\beta^i\beta^j\right]_A^{1/2}}{\left[g_{00} + 2g_{0i}\,\beta^i + g_{ij}\,\beta^i\beta^j\right]_B^{1/2}} \times \frac{1 - c\,\beta^i_B\,\frac{\partial\mathcal{T}_r}{\partial x^i_B} - \frac{\partial\mathcal{T}_r}{\partial t_B}}{1 + c\,\beta^i_A\,\frac{\partial\mathcal{T}_r}{\partial x^i_A}}, \tag{2}$$

where $\beta^i_{A/B} = \frac{1}{c}\frac{dx^i_{A/B}}{dt}$ is the coordinate velocity of $\mathcal{O}_{\mathcal{A}/\mathcal{B}}$.

C. Astrometric observables

The direction of the incident light ray observed by $\mathcal{O}_{\mathcal{B}}$ is given by the components of the spatial part of the wave vectors in the tetrad basis [5, 6]

$$n^{(i)} = -\frac{\lambda^0_{(i)} + \lambda^j_{(i)}\hat{k}_j}{\lambda^0_{(0)} + \lambda^j_{(0)}\hat{k}_j}$$

where $\hat{k}_j \equiv k_j/k_0$ with k_μ being the coordinates of the wave vector at reception (expressed in the global coordinate system). The last relation can be expressed in term of the TTF [4, 7]

$$n^{(i)} = -\frac{\lambda^0_{(i)}\left(1 - \frac{\partial\mathcal{T}_r}{\partial t_B}\right) - c\,\lambda^j_{(i)}\frac{\partial\mathcal{T}_r}{\partial x^j_B}}{\lambda^0_{(0)}\left(1 - \frac{\partial\mathcal{T}_r}{\partial t_B}\right) - c\,\lambda^j_{(0)}\frac{\partial\mathcal{T}_r}{\partial x^j_A}} \tag{3}$$

where the components of the tetrad $\lambda^\mu_{(\alpha)}$ are evaluated at $(t_B, \mathbf{x}_B)$.

III. Post-Minkowskian expansion of the TTF

The expression of the TTF as a Post-Minkoskian series is given in [2]

$$\mathcal{T}_r(\mathbf{x}_A, t_B, \mathbf{x}_B) = \frac{R_{AB}}{c} + \frac{1}{c}\sum_n \Delta^{(n)}_r(\mathbf{x}_A, t_B, \mathbf{x}_B)$$

where the superscripts (n) stand for the nth PM order (quantity of order $\mathcal{O}(G^n)$ with G the Newton gravitational constant) and $R_{AB} = |\mathbf{x}_B - \mathbf{x}_A|$.

In [4], we have shown how to compute the TTF and its derivatives up to the second PM approximation as integrals of functions depending on the metric over the Minkowskian path $z^\alpha(\mu)$ (a straight line joining the emitter and the receiver - see figure). The TTF is computed by

$$\Delta^{(1)}_r = \int_0^1 m\left[z^\alpha(\mu);\ g^{(1)}_{\alpha\beta},\ \mathbf{x}_A, t_B, \mathbf{x}_B\right]d\mu \tag{4a}$$

$$\Delta^{(2)}_r = \int_0^1\int_0^1 n\left[z^\alpha(\mu\lambda);\ g^{(2)}_{\alpha\beta},\ g^{(1)}_{\alpha\beta},\ g^{(1)}_{\alpha\beta,\gamma},\ \mathbf{x}_A, t_B, \mathbf{x}_B\right]d\lambda d\mu. \tag{4b}$$

Similarly, the derivatives of the TTF can be computed by

$$\frac{\partial\Delta^{(1)}_r}{\partial x^i_{A/B}} = \int_0^1 m_{A/B}\left[z^\alpha(\mu);\ g^{(1)}_{\alpha\beta},\ g^{(1)}_{\alpha\beta,\gamma},\ \mathbf{x}_A, t_B, \mathbf{x}_B\right]d\mu \tag{5a}$$

$$\frac{\partial\Delta^{(2)}_r}{\partial x^i_{A/B}} = \int_0^1\int_0^1 n_{A/B}\left[z^\alpha(\mu\lambda);\ g^{(2)}_{\alpha\beta},\ g^{(2)}_{\alpha\beta,\gamma},\ g^{(1)}_{\alpha\beta},\ g^{(1)}_{\alpha\beta,\gamma},\ g^{(1)}_{\alpha\beta,\gamma\delta},\ \mathbf{x}_A, t_B, \mathbf{x}_B\right]d\lambda d\mu. \tag{5b}$$

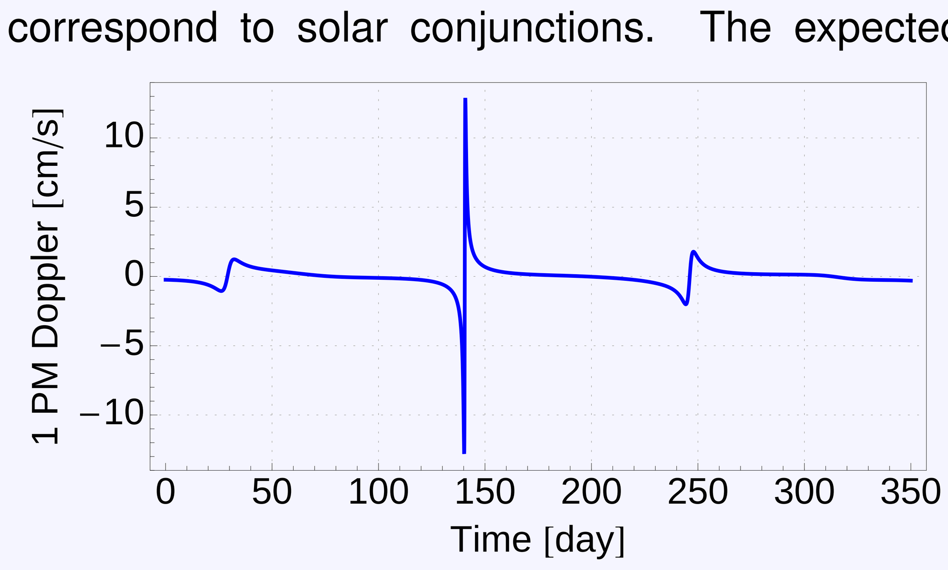
The function m, n, $m_{A/B}$ and $n_{A/B}$ are developed in details in [4]. The previous relations can be used in (1), (2) and (3).

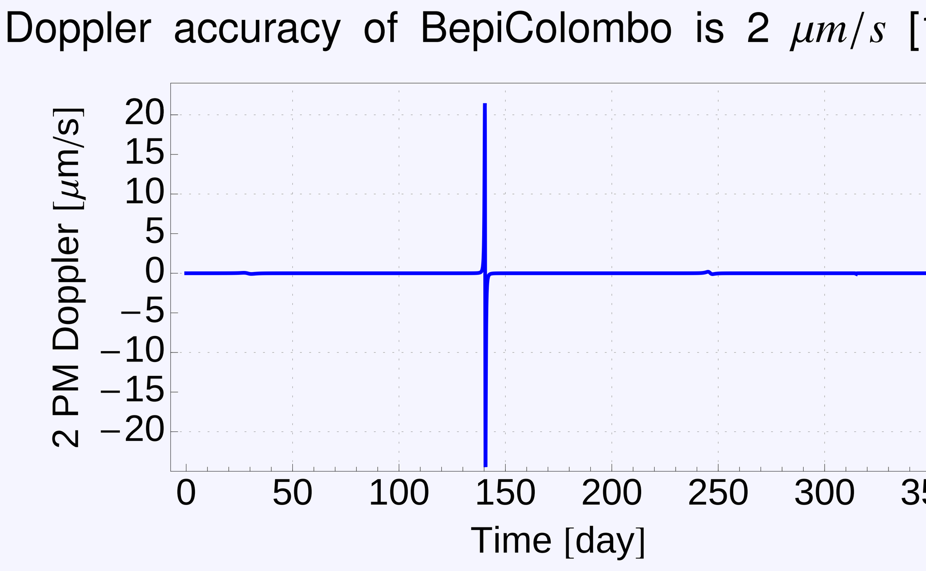
- very general formulation: no symmetry is required nor hypothesis is done.
- it can be applied to any space-time metric (in GR but also in alternative theories of gravity as long as light propagation is governed by the null geodesic equation).
- analytical validation: computation in the case of the Schwarzschild geometry performed in [4] and compared with [8].
- quite cumbersome for analytical computations but very efficient for numerical evaluations: requires only the evaluation of integrals over a straight line - easier than the determination of the full trajectory of the photon in curved space-time (a Boundary Value Problem [9]).

IV. Applications

A. Doppler link between BepiColombo and Earth

Simulation of 1 year Doppler data between an orbiter around Mercury and Earth. The three peaks correspond to solar conjunctions. The expected Doppler accuracy of BepiColombo is 2 $\mu m/s$ [10].

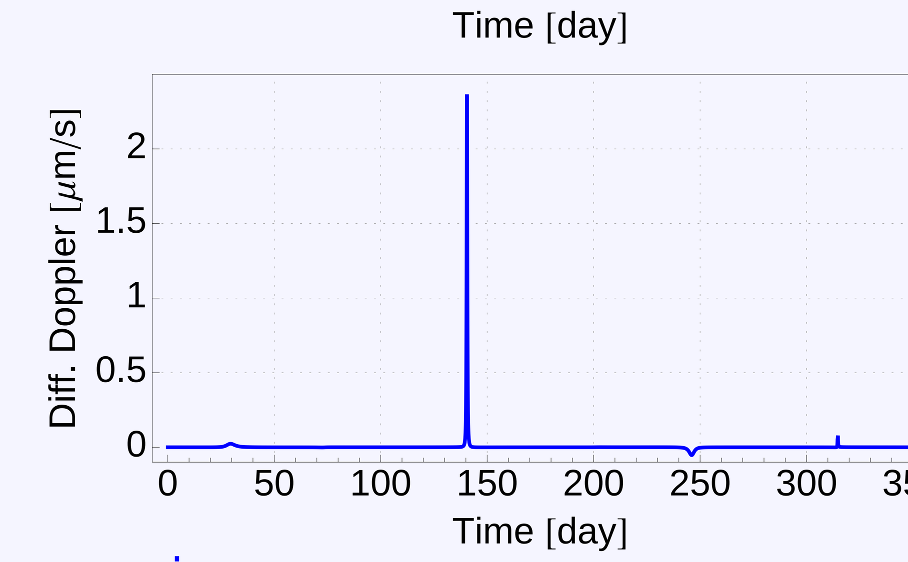




Comparison of the full 2PM approach with the approximated standard relation used in data analysis [10]

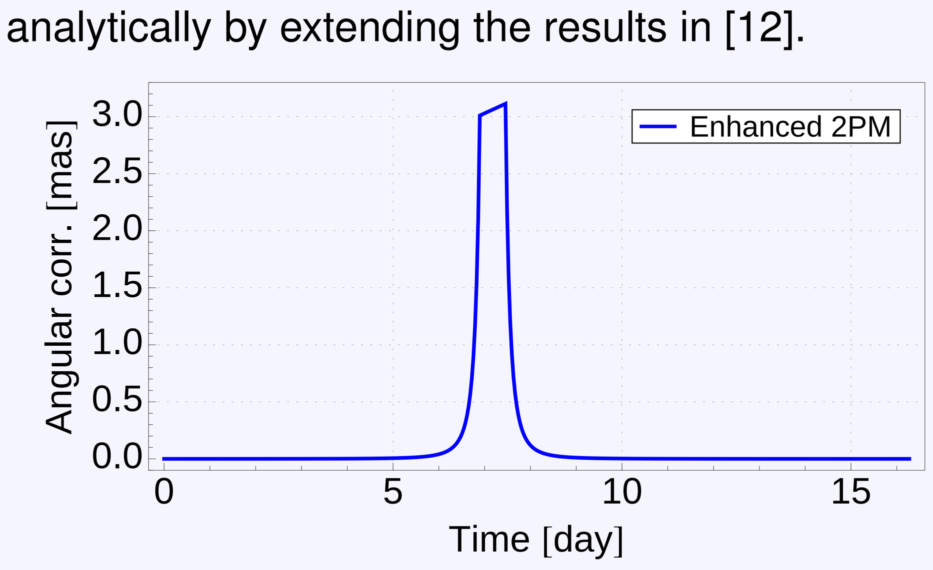
$$\mathcal{T}_r = \frac{R_{AB}}{c} + 2\frac{GM}{c^3}\ln\left[\frac{r_A + r_B + R_{AB} + 2\frac{GM}{c^2}}{r_A + r_B - R_{AB} + 2\frac{GM}{c^2}}\right], \tag{6}$$

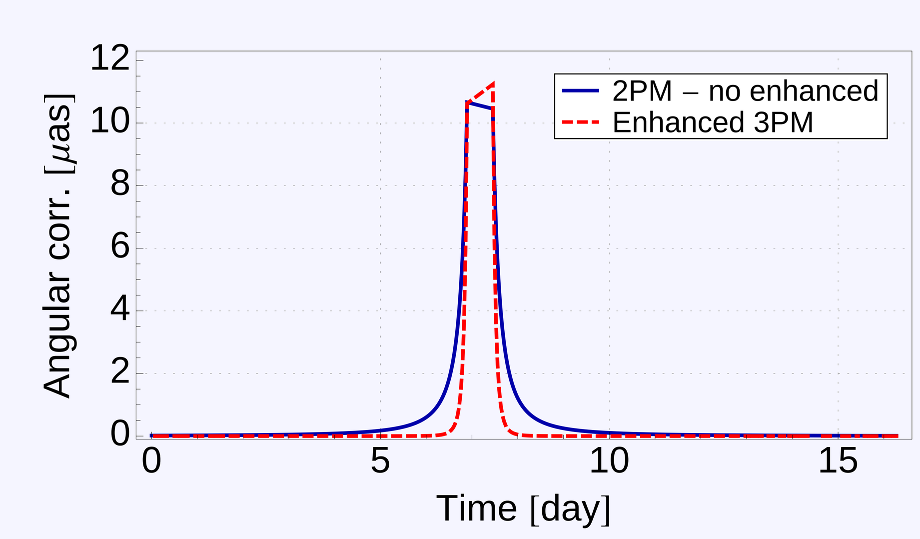
with $r_{A/B} = |\mathbf{x}_{A/B}|$.



B. Astrometric observables in a GAME-like scenario

Simulation of the angular deflection of a light ray coming from a static light source and observed by a satellite in a 1 AU orbit around the Sun (a GAME-like observation [11]) during a Solar conjunction. The expected accuracy of the GAME measurement is the μas level [11]. The 3PM term has been computed analytically by extending the results in [12].





V. Conclusion

- range, Doppler and astrometric observables can be computed as functions of the TTF and its derivatives.
- the TTF and its derivatives can be computed (up to 2PM order) by performing integrals over a straight line joining the emitter and the receiver. The integrals involve functions of the space-time metric and its derivatives only.
- powerful method in the case of numerical evaluation of the relativistic observables that can be applied to any metric (GR and alternative theories of gravity).
- method checked by considering the Schwarzschild geometry.
- applications to several future space-mission are presented.

Acknowledgments

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