Optical coordinate system for a local observer in a weak gravitational field

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We develop the method to construct the reference system of a local observer within the linear approximation on space-time curvature. Transformation to optical or Fermi coordinates is based on the solutions of the isotropic or space-like geodesic equation and parallel transport equation. The main advantage of the optical coordinates lies in their direct link with observable positions of distant objects on the celestial sphere. We also applied this method to construct the optical and Fermi coordinates for the case of accelerated and rotating observer in gravitational field induced by moving bodies of the Solar system. The transformation formulae from initial coordinates to new ones were obtained and the metric tensors in optical and Fermi frames were found in the cubic approximation on the observers velocity.

- Optical coordinates (OC) are the most appropriate for comparison with observational positions of objects on celestial sphere
- Can be easily constructed using the grav. lensing theory
- IAU recommendations: recommendations about reference frames, based on harmonic coordinates (HC)
- We develop here the exponential mapping to find the transformation formula HC=>OC (and HC=>FC Fermi coordinates), and to find a metric tensor in these coordinates
- Based on: local observer conception (based on geodesics and parallel transport equation - > Synge G. General relativity; Пирагас К., Жданов В., Александров А., Кудря Ю., Пирагас Л. Качественные и аналитические методы в релятивистской динамике., гл.3, MTW Gravity)
- Quite close to the “"observation coordinates“ by R.Maartens

FC were discussed before quite actively:

- Bahder T.// Fermi Coordinates of an Observer Moving in a Circle in Minkowski Space: Apparent Behavior of Clocks
- D. Klein, P. Collas//Exact Fermi coordinates for a class of space-times
- Ashby N., Bertotti B. Relativistic Effects in Local Inertial Frames // Phys. Rev. D. - 1986. - Vol. 34, №8. - P.2246-2259.
- Marzlin K. Fermi coordinates for a weak gravitational fields // Phys. Rev. D - 1994 - vol.50, N2, - P.888-891.
- Александров А., Жданов В., Парновский С. Релятивистская система отсчёта в околоземном пространстве и радиоинтерферомерические наблюдения // КФНТ - 1990. - Т.6, №2. - С.?
- Александров А., Жданов В., Парновский С. Релятивистские поправки в системах отсчёта локального наблюдателя при радиоинтерферометрии в космосе: Препр. / АН СССР, ин-т прикладной астрономии, №11. - Л.:1990.
- Fukushima T. The Fermi coordinates for weak gravitational fields // Cel. Mech. - 1988. - Vol. 44. - P.61-75.
- Fukushima T., Fujimoto M., Kinoshita H., Aoki S. Coordinate systems in the general relativistic framework // "Relativity in celestial mechanics and astrometry: High precision dynamical theories and observational verifications". - Leningrad (USSR). - 1985. - Proceedings of the Symposium. - P.145-168
- Жданов В., Александров А. Координаты Ферми и радиоинтерферометрические наблюдения // Вестник КУ. Астрономия - 1990. - вып. 32. - Киев, Лыбидь. - С.24-28.

OC are studied less intensively:

Александров О., Жданов В. До теорії релятивістських систем, побудованих на основі оптичних координат // Вісник КУ, Фіз.-мат. - 1992. - №3 - С.6-11.

$x^\mu{}_c(\tau)$ - observer's worldline (farther origin)=reference curve, τ is his proper time;

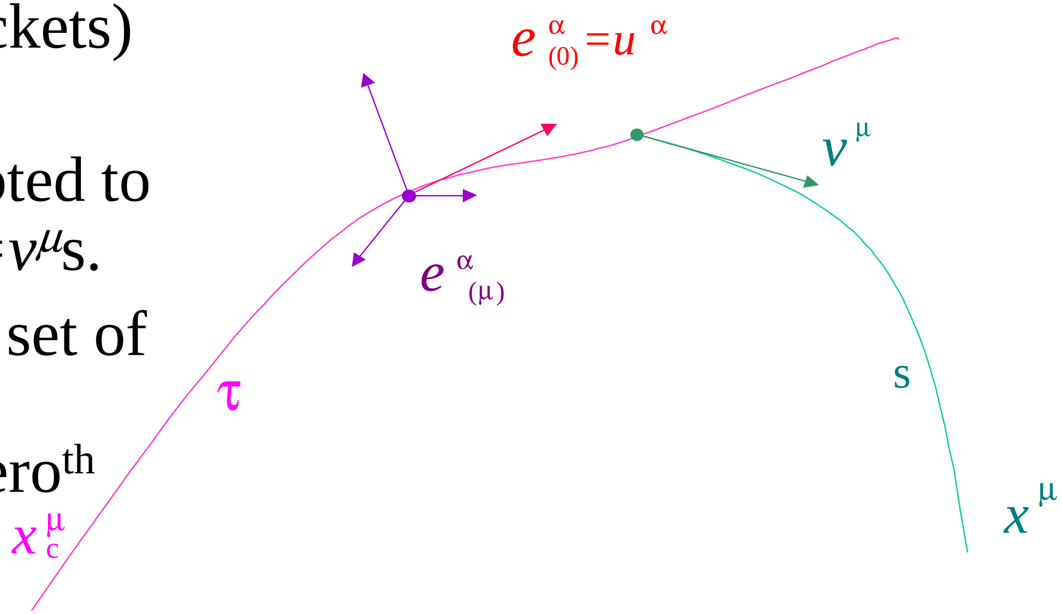
$x^\mu(\tau,s)$ – geodesics line from origin to a source with natural parameter s ($x^\mu(\tau,0)=x^\mu{}_c(\tau)$), and

v^α - tangent vector of this geodesics;

$\lambda^\mu{}_{(\alpha)}$ – set of reference vectors of the initial reference system (vector number is in brackets)

Then Riemannian normal coordinates (RNC), adopted to the reference set are: $y^\mu=v^\mu s$.

$e^\alpha{}_{(\beta)}(\tau)$ – reference vectors set of the observer (parallel transported with him), zero[th] one coincides with the observer's velocity:



$$\frac{dx_c^\mu}{d\tau}=u^\mu=e^\mu_{(0)}$$

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Riemann normal coordinates, Fermi reference system and the geodesic deviation equation

Reference sets of observer and HC are connected with each other via Lorentz transformation matrix $\Lambda_{(\alpha)(\beta)}$:

$$\lambda^{\mu}{}_{(\beta)} = \Lambda_{(\alpha)(\beta)}\, e^{\mu}{}_{(\alpha)}.$$

This set is translated parallel along the observer's worldline:

$$\frac{D\, e^{\mu}_{(i)}}{\partial\, \tau} = \Omega^{\mu}{}_{\varepsilon} e^{\varepsilon}_{(i)}$$

- Here the 4 – rotation tensor

$$\Omega_{\alpha\beta} = -a_{\alpha} u_{\beta} + a_{\beta} u_{\alpha} - \varepsilon_{\alpha\beta\mu\nu} u^{\alpha} \omega^{\beta},$$

- a^{α} - 4- acceleration,
- ω^{β} - angular 4-velocity of observer's rotation.

- Transformation formula can be found integrating the equation of the reference geodesics:

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} = 0$$

- here $\Gamma_{\alpha,\mu\nu}$ – Cristoffel's symbols, and

$$\left[\frac{dx^\mu}{ds} \right]_{s=0} = v^\mu$$

Additionally, for OC:

Geodesics is isotropic: $v^\alpha v_\alpha=0$, and for FC

Geodesics is space-like and orthogonal to the observers worldline: $u^\alpha v_\alpha=0$.

Then FC $z\alpha$ are: $z^i=y^\kappa e^i{}_{(\kappa)}$, $z^0=\tau$,

And OK $\xi\alpha$ are: $\xi^i=y^\alpha e^i{}_{(\alpha)}$, $\xi^0=\tau$.

gμν - metric tensor in the initial coordinates x^{μ}

$$g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta}$$

where h are small in comparison to 1 . Then

$$\Gamma^{\mu}_{,\nu\lambda}=\tfrac{1}{2}g^{\mu\rho}\left(g_{\rho\nu,\lambda}+g_{\rho\lambda,\nu}-g_{\nu\lambda,\rho}\right)=$$

$$=\tfrac{1}{2}\eta^{\mu\rho}\left(h_{\rho\nu,\lambda}(x^{\gamma})+h_{\rho\lambda,\nu}(x^{\gamma})-h_{\nu\lambda,\rho}(x^{\gamma})\right)$$

- Integration give us the transformation formula in linear approximation on metric

$$x^{\mu}=x^{\mu}_{c}+y^{\mu}-y^{\nu}\eta^{\mu\sigma}I_{\nu\sigma}+\tfrac{1}{2}\eta^{\mu\sigma}y^{\nu}y^{\lambda}\left(J_{\nu\lambda}-I_{\nu\lambda}\right)_{,\sigma}$$

where:

$$J_{\mu\nu}=\int_{0}^{s}\frac{f_{\mu\nu}(s')}{s'}ds'$$

$$f_{\mu\nu}=h_{\mu\nu}(x^{\alpha})-h_{\mu\nu}(x^{\alpha}{}_{c})$$

$$I_{\mu\nu}=\frac{1}{s}\int_{0}^{s}f_{\mu\nu}(s')ds'$$

FC : metric tensor

$$\overset{F}{g}_{00} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau}$$

$$\overset{F}{g}_{0i} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial y^{\lambda}} \frac{\partial y^{\lambda}}{\partial z^{i}}$$

$$\overset{F}{g}_{i\,j} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial y^{\lambda}} \frac{\partial x^{\nu}}{\partial y^{\kappa}} \frac{\partial y^{\lambda}}{\partial z^{i}} \frac{\partial y^{\kappa}}{\partial z^{j}}$$

$$\frac{\partial x^{\mu}}{\partial \tau} = u^{\mu} + \frac{\partial y^{\mu}}{\partial \tau} - y^{\nu}\eta^{\mu\sigma} \frac{\partial}{\partial \tau} I_{\nu\sigma} - \eta^{\mu\sigma} I_{\nu\sigma} \frac{\partial y^{\nu}}{\partial \tau} + \eta^{\mu\sigma} y^{\alpha} \frac{\partial y^{\beta}}{\partial \tau} \left(J_{\alpha\beta} - I_{\alpha\beta}\right)_{,\sigma}$$

$$+ \eta^{\mu\sigma} \frac{y^{\alpha} y^{\beta}}{2} \frac{\partial}{\partial \tau} \left(\dot{J}_{\alpha\beta} - I_{\alpha\beta}\right)_{,\sigma}$$

$$\frac{\partial \mathrm{y}^{\mu}}{\partial \tau} = \frac{\mathrm{D}\,\mathrm{y}^{\mu}}{\partial \tau} - \tilde{\Gamma}^{\mu}_{,\nu\lambda} \mathrm{u}^{\nu} \mathrm{y}^{\lambda}$$

$$\frac{D\, y^{\mu}}{\partial \tau} = \Omega^{\mu}{}_{\varepsilon} y^{\varepsilon}$$

$$\overset{F}{g}_{00} = \left(g_{\mu\nu}(x_c^{\gamma}) + 2\Delta_{\mu\nu} \right) u^{\mu} u^{\nu} +$$

$$+2\Omega^{\mu}{}_{\lambda} y^{\lambda} u^{\nu} \left(g_{\mu\nu}(x_c^{\gamma}) + \Delta_{\mu\nu} + \Sigma_{\mu\nu} \right) +$$

$$+\Omega^{\mu}{}_{\lambda} y^{\lambda} \Omega^{\nu}{}_{\rho} y^{\rho} \left(g_{\mu\nu}(x_c^{\gamma}) + 2\Sigma_{\mu\nu} \right)$$

$$\Delta_{\mu\nu} = \frac{1}{2} \int_0^s ds' v^{\alpha} v^{\beta} \int_0^{s'} ds'' R_{\mu\alpha\beta\nu}(s'')$$

$$\Sigma'_{\alpha\beta} = \frac{1}{2} \int_0^s \frac{ds'}{(s')^2} v^{\mu} v^{\nu} \int_0^{s'} (s'')^2 ds'' R_{\mu\alpha\beta\nu}(s'')$$

Here $R\alpha\beta\chi\delta$ - Riemannian tensor

$$2\Sigma_{\alpha\beta} = f_{\alpha\beta} - 2I_{\alpha\beta} - 2y^{\nu}\left(I_{\alpha\nu,\beta} + I_{\beta\nu,\alpha}\right) + y^{\sigma}\left(J_{\alpha\sigma,\beta} + J_{\beta\sigma,\alpha}\right) + y^{\rho} y^{\sigma}\left(J_{\sigma\rho} - I_{\sigma\rho}\right)_{,\alpha\beta}$$

$$2\Delta_{\mu\nu} = y^{\alpha} y^{\beta}\left\{K_{\alpha\beta} - J_{\alpha\beta}\right\}_{,\mu\nu} + f_{\mu\nu}(s) - 2y^{\alpha}\left\{J_{\alpha(\mu,\nu)} - \tilde{\Gamma}_{\alpha,\mu\nu}\right\}$$

$$K_{\alpha\beta} = s\int_0^s \frac{ds'}{(s')^2}\left(f_{\alpha\beta}(s') - F_{\alpha\beta,\varsigma}(x_c^\gamma,0)v^\varsigma s'\right)$$

- Taking into account that:

$$\frac{\partial\, y^\beta}{\partial\, z^i} = e_{(i)}^\beta$$

$$\overset{F}{g}_{0i} = \left\{(g_{\alpha\beta}(x_c^\gamma) + \Sigma_{\alpha\beta} + \Delta_{\alpha\beta})u^\alpha + \left(g_{\mu\beta}(x_c^\gamma) + 2\Sigma_{\mu\beta}\right)\Omega^\mu{}_\nu y^\nu\right\}e_{(i)}^\beta$$

$$\overset{F}{g}_{ij} = -\delta_{ij} + 2\Sigma_{\alpha\beta}e_{(i)}^\alpha e_{(i)}^\beta$$

OC:

- metric transformation:

$$\overset{O}{g}_{00} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau}$$

$$\overset{O}{g}_{0i} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial y^{\lambda}} \frac{\partial y^{\lambda}}{\partial \xi^{i}}$$

$$\overset{O}{g}_{i\,j} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial y^{\lambda}} \frac{\partial x^{\nu}}{\partial y^{\kappa}} \frac{\partial y^{\lambda}}{\partial \xi^{i}} \frac{\partial y^{\kappa}}{\partial \xi^{j}}$$

Metric tensor in OC:

$$\overset{O}{g}_{ij} = -\delta_{ij} + v^i v^j + 2\Sigma_{\mu\nu} e^{\mu}_{(i)} e^{\nu}_{(j)} +$$
$$+2\Sigma_{\mu\nu} u^{\mu} u^{\nu} v^i v^j - \Sigma_{\mu\nu}\left(u^{\mu} e^{\nu}_{(j)} v^i + u^{\nu} v^j e^{\mu}_{(i)}\right)$$

$$\overset{O}{g}_{0i} = \left(g_{\mu\beta}(x_c^{\gamma}) + 2\Sigma_{\mu\beta}\right) \Omega^{\mu}{}_{\nu} y^{\nu} e^{\beta}_{(i)} +$$
$$\left(g_{\alpha\beta}(x_c^{\gamma}) + \Sigma_{\alpha\beta} + \Delta_{\alpha\beta}\right)\left(e^{\beta}_{(i)} - u^{\beta}\frac{\xi^i}{\xi}\right)u^{\alpha}$$

$$\overset{O}{g}_{00} = \left(g_{\mu\nu}(x_c^{\gamma}) + 2\Delta_{\mu\nu}\right)u^{\mu}u^{\nu} +$$
$$+2\Omega^{\mu}{}_{\lambda} y^{\lambda} u^{\nu}\left(g_{\mu\nu}(x_c^{\gamma}) + \Delta_{\mu\nu} + \Sigma_{\mu\nu}\right) +$$
$$+\Omega^{\mu}{}_{\lambda} y^{\lambda} \Omega^{\nu}{}_{\rho} y^{\rho}\left(g_{\mu\nu}(x_c^{\gamma}) + 2\Sigma_{\mu\nu}\right)$$

An example: moving point masses

- Initial metric:

$$g_{00} = 1 - 2\varphi$$

$$g_{ik} = -\delta_{ik}\left(1 + 2\varphi\right)$$

$$g_{0i} = -4\pi^{i}\varphi$$

- where π^{i} is a velocity, m - mass
- And gravitational potential is:

$$\varphi(\vec{x}) = \frac{m}{\left|\vec{x} - \vec{x}_{p}\right|}$$

Transformation formula:

$$\vec{x} = \vec{x}_c + \vec{y} + m\left(\frac{1}{p} - \frac{U}{y}\right)\left[\vec{y}\left(1 - \frac{(y^0)^2}{\vec{y}^2}\right) + 4y^0\vec{\varpi}\right] - \vec{D}\frac{(y^0)^2 + \vec{y}^2 + 4y^0(\vec{\pi}\vec{y})}{\vec{y}^2}$$

$$x^0 = x_c^0 + y^0 - 2m\left(\frac{1}{p} - \frac{U}{y}\right)(y^0 + 2(\vec{\pi}\vec{y}))$$

$$U = \ln\left|\frac{Ry + (\vec{R}\vec{y})}{py + (\vec{p}\vec{y})}\right| \qquad \vec{D} = m\frac{\vec{d}}{(d)^2}\left(pR - (\vec{p}\vec{R})\right)$$

$$\vec{\varpi} = \vec{\pi} - \frac{\vec{y}}{\vec{y}^2}(\vec{\pi}\vec{y}) \qquad \vec{R} = \vec{x} - \vec{x}_p \qquad \vec{d} = \vec{p} - \frac{(\vec{p}\vec{v})}{(\vec{v})^2}\vec{v}$$

ОС:

$$\overset{O}{g}_{00} = 1 + 2m\left(\frac{1}{p} - \frac{1}{R_0} - \frac{(\vec{p}\vec{\xi})}{\left|\vec{p}\right|^3} + \beta\xi\left(\frac{(\vec{n}\vec{p})}{\left|\vec{p}\right|^3} - \frac{(\vec{n}\vec{R}_0)}{\left|\vec{R}_0\right|^3}\right)\right)$$

$$\vec{R} = \vec{p} + \vec{\xi} - \xi\,\vec{u} = \vec{R}_0 - \xi\,\vec{u}$$

$$\overset{O}{g}_{ij} = -\overset{O}{F}\sigma_{ij} + 2\Sigma_{00}v^i v^j - \left(\Sigma_{oi}v^j + \Sigma_{oj}v^i\right)$$

$$\overset{O}{F} = 1 + 2m\left[-\frac{U}{y} + \frac{p(\vec{R}\vec{y}) - R(\vec{p}\vec{y})}{(yd)^2}\right] \qquad \Sigma_{00} = m\left[\frac{1}{p} + \frac{1}{R} - 2\frac{U}{y}\right]$$

$$\Sigma_{0i} = -4\pi^i\Sigma_{00} + \frac{2m\xi^i}{(\xi)^3}(\vec{\pi}\vec{\xi})\left[2U - \frac{p(\vec{R}\vec{\xi}) - R(\vec{p}\vec{\xi})}{\xi pR}\right] - 2\frac{(\vec{\pi}\vec{\xi})}{(\xi)^2}\left(\frac{1}{p} - \frac{1}{R}\right)D^i$$

$$\overset{O}{g}_{0i} = \overset{O}{A} e^{0}_{(i)} + \overset{O}{B} z^{i} + \overset{O}{C}{}^{i} + \overset{O}{D} \pi^{i} - \overset{O}{E} \lambda^{i}$$

$$\overset{O}{A} = \gamma - m\gamma \left[\frac{2U}{y} - \frac{(\vec{p}\vec{y})}{(p)^{3}} \right] \qquad \overset{O}{C}{}^{i} = 2(\vec{\pi}\vec{\xi}) \left\{ \frac{md^{i}}{p^{3}} - \frac{D^{i}}{R(\xi)^{2}} \right\}$$

$$\overset{O}{B} = -\overset{O}{A} - 2\beta^{2} + 2m \frac{(\vec{\pi}\vec{\xi})}{\xi} \left\{ \frac{1}{R} - \frac{U}{\xi} + \frac{(\vec{p}\vec{\xi})}{(p)^{3}} \right\}$$

$$\overset{O}{D} = 2m \left\{ 2\left(\frac{U}{\xi} - \frac{1}{p} - \frac{1}{R} \right) - \frac{(\vec{p}\vec{\xi})}{p^{3}} \right\}$$

$$\overset{O}{E} = \frac{U}{\xi} - \frac{(\vec{R}\vec{\xi})}{(d\xi)^{2}} (p - R)$$

Thank You for Your patience!