On the definition of a reference frame and the associated space in a general spacetime

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Abstract

A reference frame F can be defined as an equivalence class of spacetime charts (coordinate systems) having a common domain U and exchanging by a spatial coordinate change. The associated physical space is made of the world lines having constant space coordinates in any chart of the class. The unit tangent vector to these lines defines in U a 4-velocity field v_F. These are local definitions. The data of a global 4-velocity field v defines a global "reference fluid". The associated global physical space is made of the maximal integral lines of that vector field. Assume that the restriction of the global 4-velocity field v to the domain U is v_F. In that case, the local space can be identified with a part (a submanifold) of the global space.

Introduction

A reference frame is essentially a three-dimensional network of observers equipped with clocks and meters. To any reference frame, one should be able to associate some three-dimensional *space,* in which the observers of the network are at rest (even though their mutual distances may depend on time). Clearly, both notions are fundamental ones for physics. In the relativistic theories of gravitation, the spacetime metric tensor $g_{\mu\nu}$ is a field. Rigid reference frames are not relevant. The relevant notion is that of a *reference fluid,* given by a 4-velocity field v on spacetime [1]: v is the unit tangent vector field to the world lines of the observers belonging to the network. In standard practice, one often admits implicitly that a reference frame can be fixed by the data of one *coordinate system* (or *chart*). The link with the definition by the 4-velocity field v tangent to a network of observers is as follows [1]. Any admissible chart on the spacetime, $\chi : X \mapsto (x^\mu)$ $(\mu = 0, ..., 3)$, defines a unique network of observers, whose world lines are

$$x^j = \text{Constant } (j = 1, 2, 3),\ \ x^0 \text{ variable.} \tag{1}$$

The corresponding four-velocity field v has the following components in the chart χ:

$$v^0 \equiv \frac{1}{\sqrt{g_{00}}}, \qquad v^j = 0 \quad (j = 1, 2, 3). \qquad [\text{signature } (+ - - -)] \tag{2}$$

This is valid only within the domain of definition U of the chart χ, thus in general not in the whole spacetime.

The notion of the space associated with a network of observers was missing in the general-relativistic literature. But in practice, one cannot dispense with some notion of a *physical space*. One needs to define the spatial positions of physical objects, even though these depend on the reference network considered. One also needs a physical space to define the quantum space of states, and spatial vectors or tensors such as the usual 3-velocity vector or the rotation rate tensor of a triad. We recall the results obtained previously [2] regarding the definition of a local reference frame and the associated space. Then we announce results of a current work, that aims at defining global notions and at relating them to the formerly introduced local notions.

A local definition of a reference frame F and the associated space $\mathrm{M_F}$

One may formally define a reference frame as being an *equivalence class of charts* which are all defined on a given open subset U of the spacetime V and are related two-by-two by a *purely spatial* coordinate change:

$$x'^0 = x^0, \quad x'^k = \phi^k((x^j)) \qquad (j, k = 1, 2, 3). \tag{3}$$

This does define an equivalence relation [2]. Thus a reference frame F, i.e. an equivalence class for this relation, can indeed be given by *the data of one chart* $\chi : X \mapsto (x^\mu)$ *with its domain of definition* U (an open subset of the spacetime manifold V). Namely, F is the equivalence class of (χ, U). I.e., F is the set of the charts χ' which are defined on U, and which are such that the transition map $f \equiv \chi' \circ \chi^{-1} \equiv (\phi^\mu)$ corresponds with a purely spatial coordinate change (3).

The associated (local) space $\mathrm{M_F}$ and its manifold structure. Applications

The former definition has physical meaning, since the world lines (1) and the 4-velocity field (2) are invariant under the purely spatial coordinate changes (3). The local physical space $\mathrm{M} = \mathrm{M_F}$ is mathematically defined as the set of the world lines (1). In detail: let $P_S : \mathsf{R}^4 \rightarrow \mathsf{R}^3$, $\quad \mathbf{X} \equiv (x^\mu) \mapsto \mathbf{x} \equiv (x^j)$, be the spatial projection. A world line l is an element of the set $\mathrm{M_F}$ iff there is a chart $\chi \in \mathrm{F}$ and a triplet $\mathbf{x} \equiv (x^j) \in \mathsf{R}^3$, such that l is the set, assumed non-empty, of *all* points X in the domain U, whose spatial coordinates are $\mathbf{x}$:

$$l = \{\, X \in \mathrm{U};\ P_S(\chi(X)) = \mathbf{x} \,\} \quad \text{and } l \neq \emptyset. \tag{4}$$

Consider a chart $\chi \in \mathrm{F}$. With any world line $l \in \mathrm{M_F}$, let us associate the triplet $\mathbf{x} \equiv (x^j)$ made with the *constant* spatial coordinates of the points $X \in l$. We thus define a mapping

$$\widetilde{\chi} : \mathrm{M_F} \rightarrow \mathsf{R}^3, \quad l \mapsto \mathbf{x} \text{ such that } \forall X \in l,\ \chi^j(X) = x^j \ (j = 1, 2, 3). \tag{5}$$

Through Eq. (4), the world line $l \in \mathrm{M_F}$ is determined uniquely by the data $\mathbf{x}$. I.e., the mapping $\widetilde{\chi}$ is one-to-one. Consider the set $\mathcal{T}$ of the subsets $\Omega \subset \mathrm{M_F}$ such that,

$$\forall \chi \in \mathrm{F}, \quad \widetilde{\chi}(\Omega) \text{ is an open set in } \mathsf{R}^3. \tag{6}$$

We showed that $\mathcal{T}$ is a topology on $\mathrm{M_F}$, and that the set of the mappings $\widetilde{\chi}$ defines a structure of differentiable manifold on that topological space $(\mathrm{M_F}, \mathcal{T})$: *The spatial part of any chart* $\chi \in \mathrm{F}$ *defines a chart* $\widetilde{\chi}$ *on* $\mathrm{M_F}$ [2].

A Hamiltonian operator of relativistic QM depends *precisely* [3] on the reference frame F as defined above. The Hilbert space $\mathcal{H}$ of quantum-mechanical states is the set of the square-integrable functions defined on the associated *space manifold* $\mathrm{M_F}$ [4]. Prior to this definition, $\mathcal{H}$ depended on the particular spatial coordinate system. This does not seem acceptable. Also, the full algebra of spatial tensors can now be defined in a simple way: a spatial tensor field is simply a tensor field on the space manifold $\mathrm{M_F}$ associated with a reference frame F [5]. A simple example is the *3-velocity* of a particle (or a volume element) in a reference frame: this is a spatial vector, i.e., the current 3-velocity at an event $X \in \mathrm{U}$ is an element of the tangent space at $x(X) \in \mathrm{M_F}$.

The global space N_v associated with a time-like vector field v

The former definitions of a *reference frame* and the associated space manifold apply to a domain U of the spacetime V, such that at least one regular chart can be defined over the whole of U. Thus these are *local* definitions. Can the definition of a *reference fluid* by the data of a *global* four-velocity field v lead to a global notion of space? If yes, what is the link with the former local notions?

Given a global vector field v on the spacetime V, and given an event $X \in \mathrm{V}$, let C_X be the solution of

$$\frac{dC}{ds} = v(C(s)), \qquad C(0) = X \tag{7}$$

that is defined on the *largest possible* open interval I_X containing 0 [6]. Call the *range* $l_X \equiv C_X(\mathrm{I}_X) \subset \mathrm{V}$ the "maximal integral line at X". If $X' \in l_X$, then it is easy to show that $l_{X'} = l_X$. The *global space* N_v *associated with the vector field* v is the set of the maximal integral lines of v:

$$\mathrm{N}_v \equiv \{l_X;\ X \in \mathrm{V}\}. \tag{8}$$

Local existence of adapted charts and manifold structure of the global set N_v

A chart χ with domain $\mathrm{U} \subset \mathrm{V}$ is said "v–adapted" iff the spatial coordinates remain constant on any integral line l of v — more precisely, remain constant on $l \cap \mathrm{U}$: For any $l \in \mathrm{N}_v$, there is some $\mathbf{x} \equiv (x^j) \in \mathsf{R}^3$ such that

$$\forall X \in l \cap \mathrm{U}, \quad P_S(\chi(X)) = \mathbf{x}. \tag{9}$$

For any v–adapted chart χ, the mapping

$$\bar{\chi} : l \mapsto \mathbf{x} \text{ such that (9) is verified} \tag{10}$$

is well defined on

$$\mathrm{D_U} \equiv \{l \in \mathrm{N}_v;\ l \cap \mathrm{U} \neq \emptyset\}. \tag{11}$$

Call the v–adapted chart χ "nice" if the mapping $\bar{\chi}$ is one-to-one. We are making progress towards proving the following: Assume the global vector field v on V is non-vanishing. Then (perhaps under some additional assumption regarding the manifold V and/or the field v), for any point $X \in \mathrm{V}$, there exists a nice v–adapted chart χ whose domain is an open neighborhood of X.

Consider the set $\mathcal{F}_v$ made of all nice v–adapted charts on the spacetime manifold V, and consider the set $\mathcal{A}$ made of the mappings $\bar{\chi}$, where $\chi \in \mathcal{F}_v$, Eq. (10). A such mapping $\bar{\chi}$ is defined on the set $\mathrm{D_U}$ — a subset of the three-dimensional "space" N_v, Eq. (11). (Here U is the domain of the v–adapted chart $\chi \in \mathcal{F}_v$.) If the above conjecture is true, then we can define a topology on the global space N_v, similarly with (6). We can then also prove that $\mathcal{A}$ is an atlas on that topological space, thus defining a structure of differentiable manifold on the global set N_v. In order to prove the latter result, the main thing we prove is the compatibility of any two charts $\bar{\chi}$, $\bar{\chi}'$ on N_v, associated with two nice v-adapted charts $\chi, \chi' \in \mathcal{F}_v$. This is less easy than in the local case [2], because two v-adapted charts χ and χ' have in general different domains U and U', and we may have

$$\mathrm{U} \cap \mathrm{U}' = \emptyset, \quad l \cap \mathrm{U} \neq \emptyset, \quad l \cap \mathrm{U}' \neq \emptyset. \tag{12}$$

I.e., the domains of the maps $\bar{\chi}$ and $\bar{\chi}'$ do overlap, although the domains of the charts χ and χ' do not.

The local space $\mathrm{M_F}$ is a submanifold of the global space N_v

Let v be a non-vanishing vector field on V, and let F be a reference frame *made of nice v–adapted charts,* all defined on the same open set $\mathrm{U} \subset \mathrm{V}$.

Let $l \in \mathrm{M_F}$, thus there is some chart $\chi \in \mathrm{F}$ and some $\mathbf{x} \in \mathsf{R}^3$ such that $l = \{\, X \in \mathrm{U};\ P_S(\chi(X)) = \mathbf{x} \,\}$. Then, for any $X \in l$, we have $l' \equiv l_X \in \mathrm{N}_v$ and $l = l' \cap \mathrm{U}$. We have moreover $l' = \bar{\chi}^{-1}(\mathbf{x}) = \bar{\chi}^{-1}(\widetilde{\chi}(l))$. Hence, the mapping $I : \mathrm{M_F} \rightarrow \mathrm{N}_v$, $l \mapsto l'$ is just $I = \bar{\chi}^{-1} \circ \widetilde{\chi}$. This is a one-to-one mapping of $\mathrm{Dom}(\widetilde{\chi}) = \mathrm{M_F}$ onto $\mathrm{Dom}(\bar{\chi}) = \mathrm{D_U}$. Thus the local space $\mathrm{M_F}$ is made of the intersections with the local domain U of the world lines belonging to the global space N_v, and we may identify $\mathrm{M_F}$ with the subspace $I(\mathrm{M_F}) = \mathrm{D_U}$ of N_v.

Conclusion

We proposed to define a *reference frame* as being an equivalence class of spacetime charts χ which have a common domain U and which exchange two-by-two by a purely spatial coordinate change. This definition is practical, because it gives a methodology to use coordinate systems in a consistent and physically meaningful way: the data of one spacetime coordinate system (x^μ) defines (in its domain of definition U) the 4-velocity field of a network of observers, Eq. (2). The coordinate systems that exchange with (x^μ) by a purely spatial coordinate change (3) belong to the same reference frame and indeed the associated 4-velocity field (2) is the same. Using a general coordinate change instead, allows us to go to any other possible reference frame.

A precise notion of a *physical space* did not exist before in a general spacetime, to our knowledge. We defined two distinct concepts: a local one and a global one, which however are intimately related together. In either case, the space is the set of the world lines that belong to the given (local) reference frame, respectively to the given (global) reference fluid:

i) Consider a (local) reference frame in the specific sense meant here, i.e. a set F of charts, all defined on the same subdomain U of the spacetime, and exchanging by a change of the form (3). This allows one to define a "local space" $\mathrm{M_F}$: this is the set of the world lines (1) [more precisely the set of the world lines (4)] [2]. Each of these world lines is included in the common domain U of all charts $\chi \in \mathrm{F}$.

ii) The data of a (global) reference fluid, i.e. a global non-vanishing 4-vector field v, allows one to define a "global space" N_v: this is the set of the maximal integral lines of v.

Both of the local space $\mathrm{M_F}$ and the global space N_v can be endowed with a structure of differentiable manifold (if some conjecture is true, for the global space), essentially because each one is endowed with a specific set of charts: the charts $\widetilde{\chi}$, where χ is any chart in some reference frame (or equivalence class) F, in case (i). And the charts $\bar{\chi}$, where χ is any chart which is *adapted* to some global vector field v and such that $\bar{\chi}$ is one-to-one, in case (ii). The manifold structure has a practical aspect: Locally, the position of a point in the space can be specified by different sets of spatial coordinates, which exchange smoothly: $x'^k = \phi^k((x^j)) \quad (j, k = 1, 2, 3)$, and we may use standard differential calculus for mappings defined on that space, by choosing any such coordinates. This applies to both the local space $\mathrm{M_F}$ and the global space N_v.

There is a close link between the local space $\mathrm{M_F}$ and the global space N_v, provided the 3D network of observers is indeed the same in the two cases. [I.e.: provided the 4-velocity field (2) associated with the reference frame F is the restriction of v to the common domain U of all charts $\chi \in \mathrm{F}$.] If that is true, one may associate with each world line $l \in \mathrm{M_F}$ the world line $l' \in \mathrm{N}_v$, of which l is just the intersection with the domain U. Thus the local space can be identified with a part (a submanifold) of the global space.

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