Space-Time reference systems

Michael Soffel

Dresden Technical University

What are Space-Time Reference Systems (RS)?

Science talks about events (e.g., observations):

something that happens during a short interval of time
in some small volume of space

Both space and time are
considered to be continous

combined they are described
by a ST manifold

A Space-Time RS basically is a ST coordinate system (t,x) describing the ST position of events in a certain part of space-time

In practice such a coordinate system has to be realized in nature with certain observations; the realization is then called the corresponding

Reference Frame

Newton‘s space and time

In Newton's ST things are quite simple:

Time is absolute as is Space ->

there exists a class of preferred inertial coordinates (t,**x**) that have direct physical meaning, i.e.,

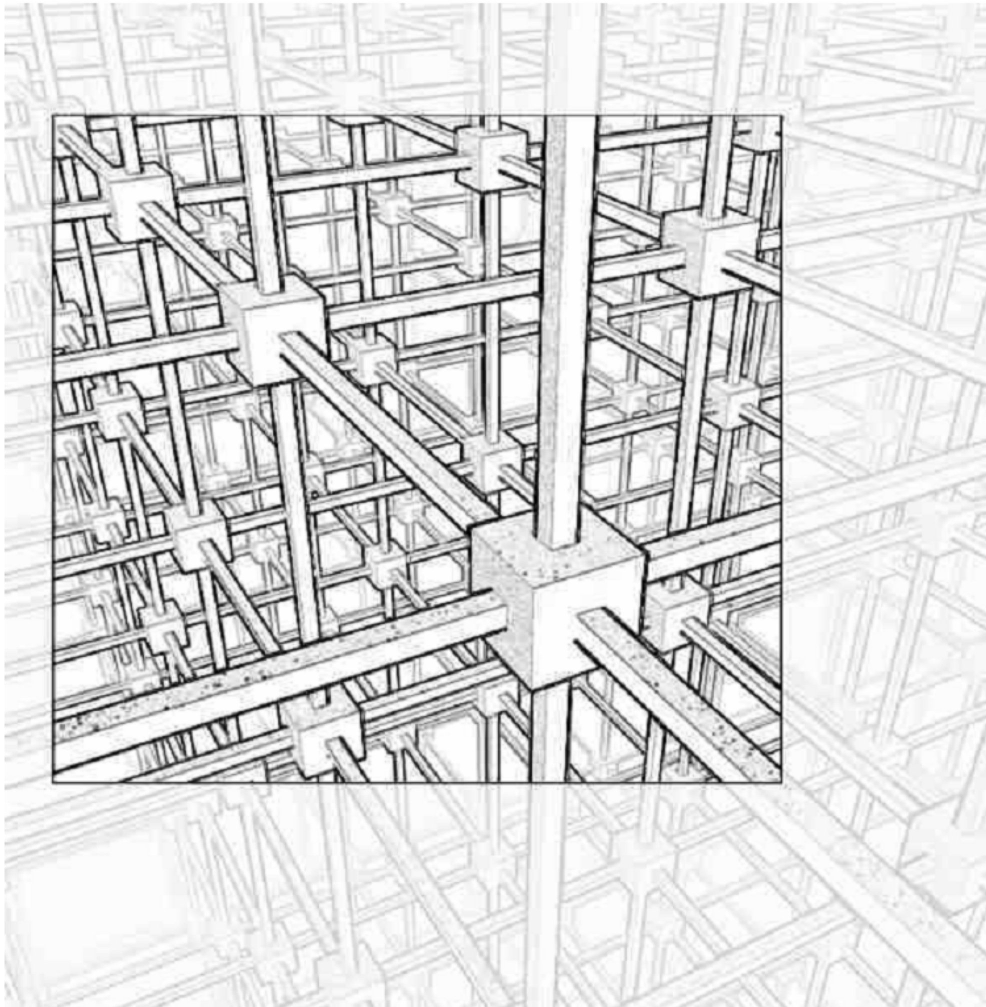
observables can be obtained directly from the coordinate picture of the physical system.

Example: $\Delta\tau = \Delta t$

(proper) time indicated by some clock

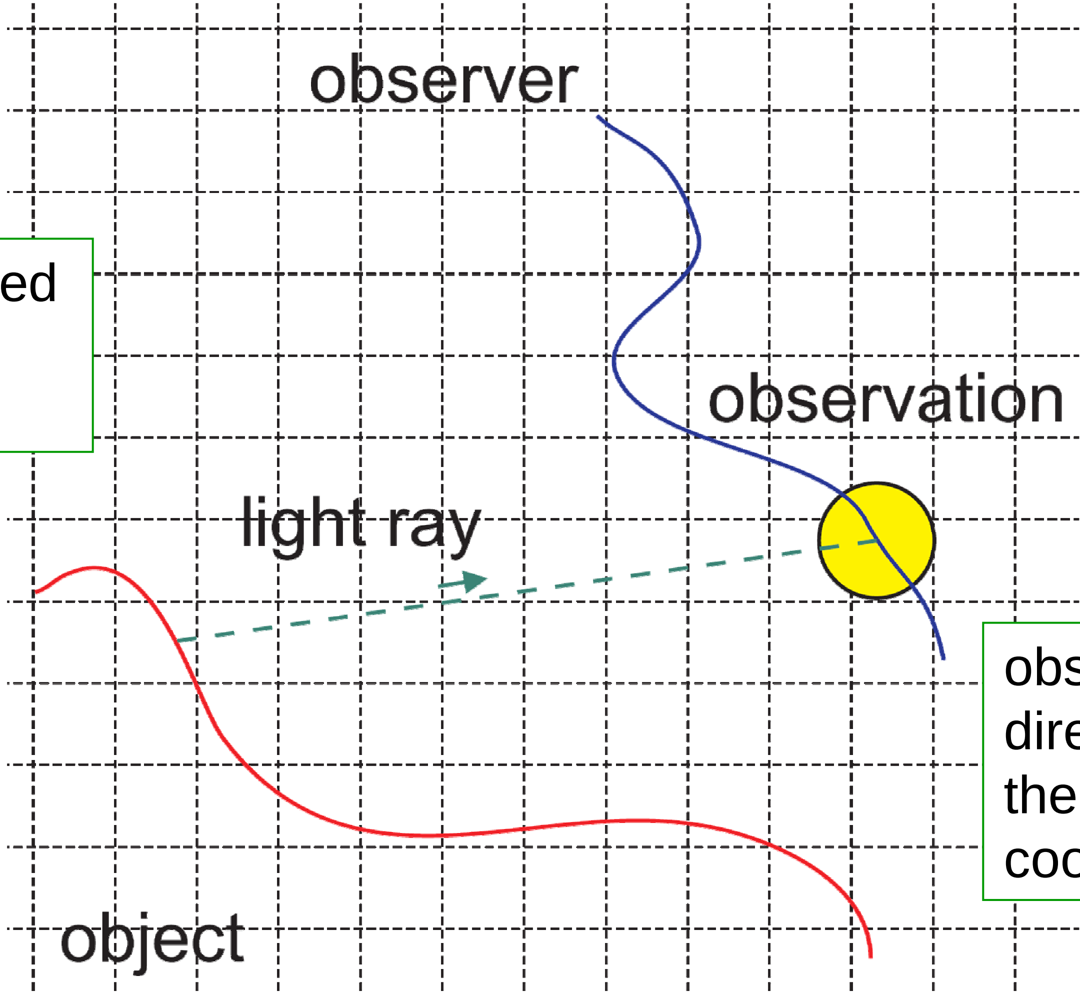
coordinate time

Newton's absolute space



Newtonian astrometry

physically preferred global inertial coordinates



observables are directly related to the inertial coordinates

Example: observed angle θ between two incident light-rays 1 and 2

$$\mathbf{x}_i(t) = \mathbf{x}_i^0 + c\,\mathbf{n}_i\,(t - t_0)$$

$$\cos\theta = \mathbf{n}_1 \cdot \mathbf{n}_2$$

Is Newton‘s conception of Time and Space in accordance with nature? NO!

The reason for that is related with properties of light propagation upon which time measurements are based

Presently: The (SI) second is the duration of

9 192 631 770

periods of radiation that corresponds to a certain transition in the Cs-133 atom

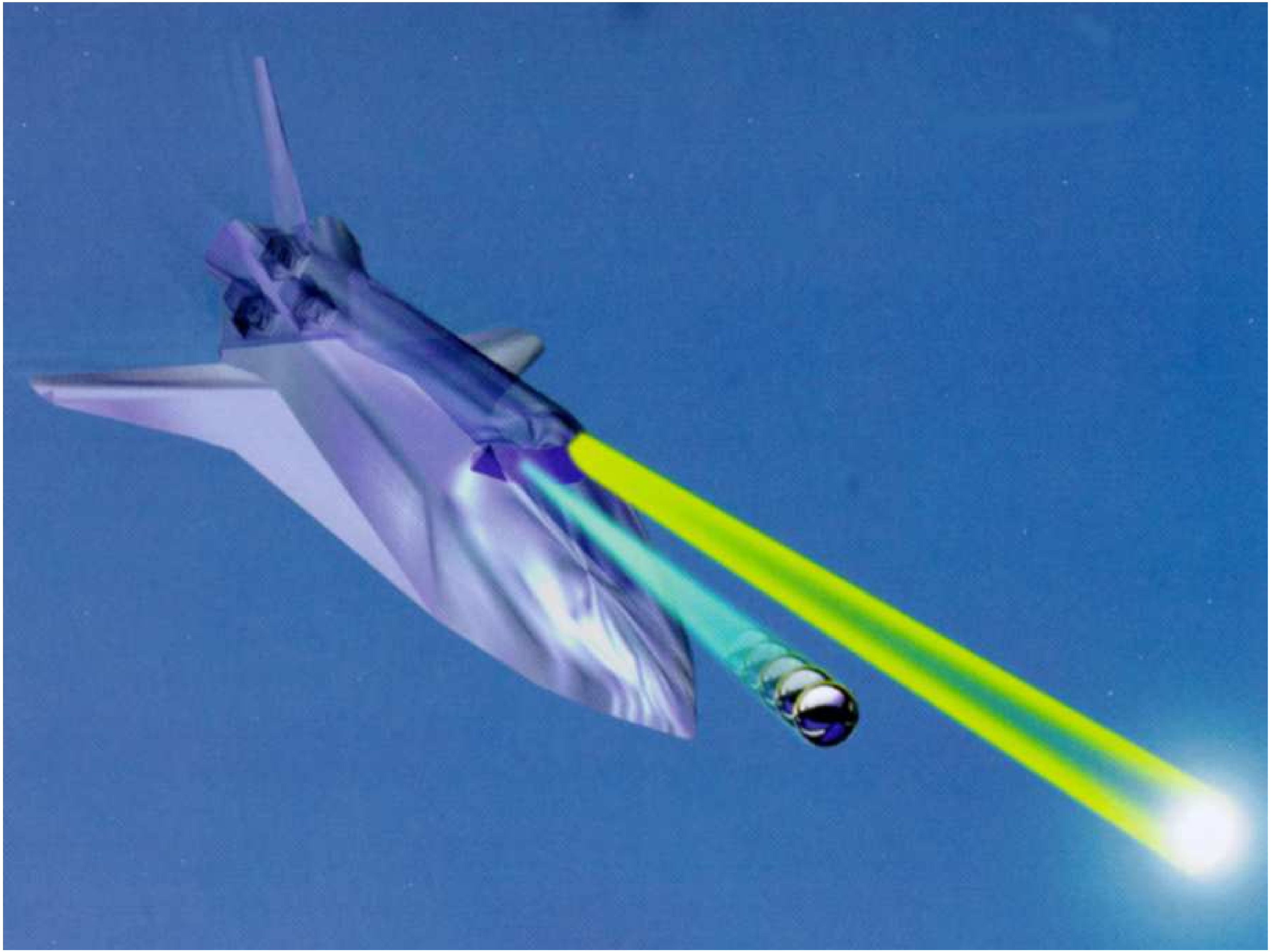
Principle of the constancy of the velocity of light in vacuum:

The light velocity is independent upon the state of motion of the light source, frequency and polarization

$$c' = c + k\, v$$

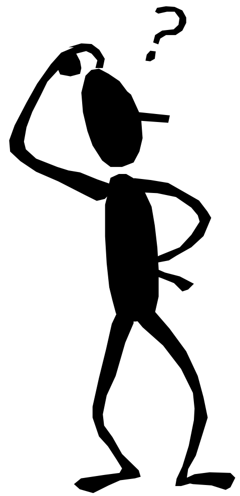
If violated:

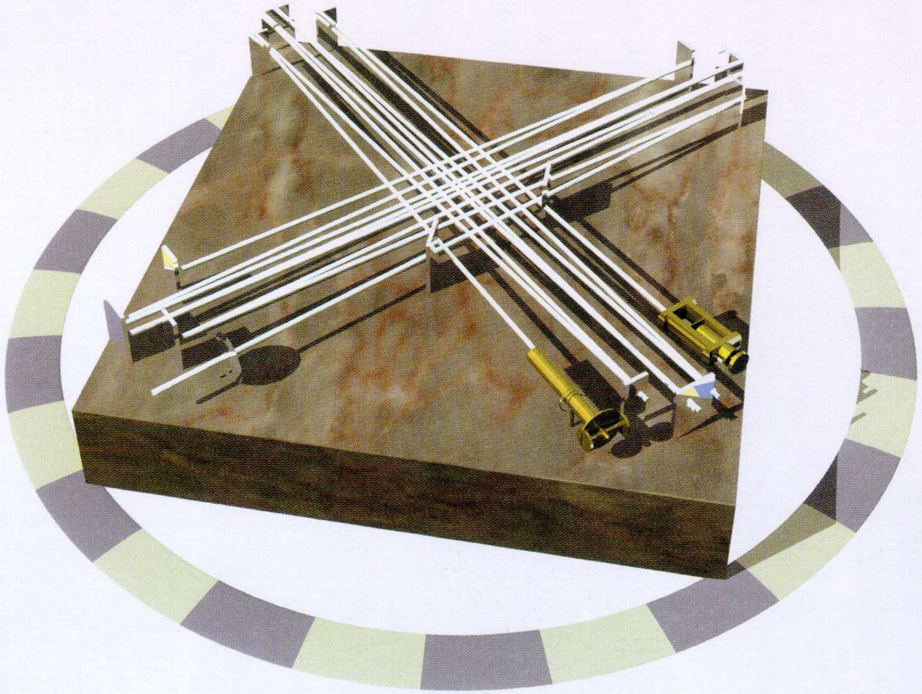
up to 5 images could be seen of a star in a binary system at the same time



Michelson und Morley (1887):

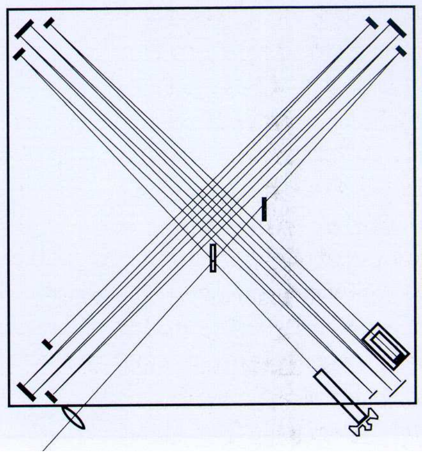
the value of c also does not depend upon the velocity of the observer





MESSUNG VON UNTERSCHIEDEN IN DER LICHTGESCHWINDIGKEIT
Im Michelson-Morley-Interferometer wird das Licht einer Quelle durch eine halbverspiegelte Glasscheibe in zwei Strahlen aufgeteilt. Das Licht der beiden Strahlen bewegt sich rechtwinklig zueinander und wird am Ende wieder zu einem einzigen Strahl vereinigt, indem es abermals zu der halbverspiegelten Scheibe gelenkt wird. Je nach Strahllänge und nach der Lichtgeschwindigkeit in den beiden Strahlen überlagern sich diese in unterschiedlicher Weise: Trifft Wellenberg auf Wellenberg, verstärken sich die Wellen gegenseitig, trifft Wellenberg auf Wellental, löschen sich die Teilstrahlen aus. Veränderungen, etwa der Übergang von Auslöschung zu Verstärkung, lassen sich beobachten und zeigen an, wenn die relative Lichtgeschwindigkeit in den Teilstrahlen variiert.

Rechts: Diagramm des Experiments nach der Abbildung, die 1887 im Scientific American erschien.

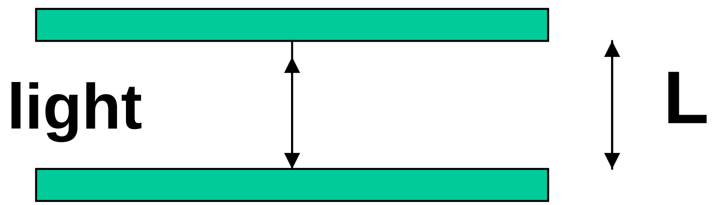


The famous Michelson & Morley experiment 1887

Cleveland, Ohio

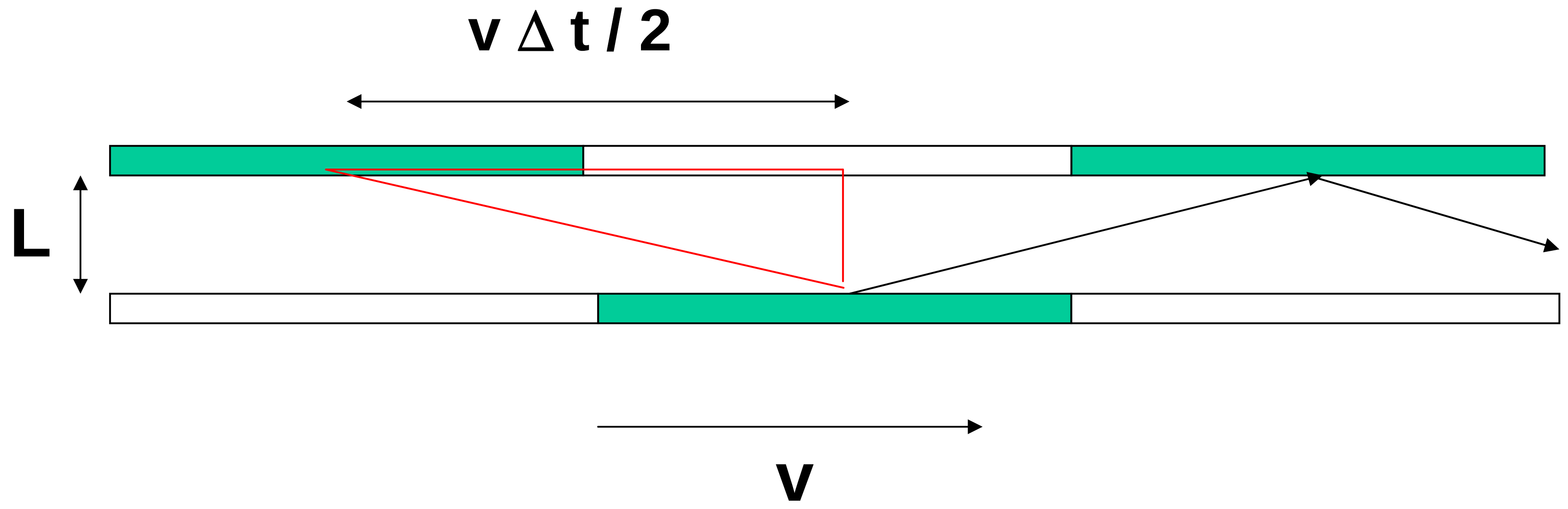
turning the apparatus did not change the interference pattern

A light-clock at rest



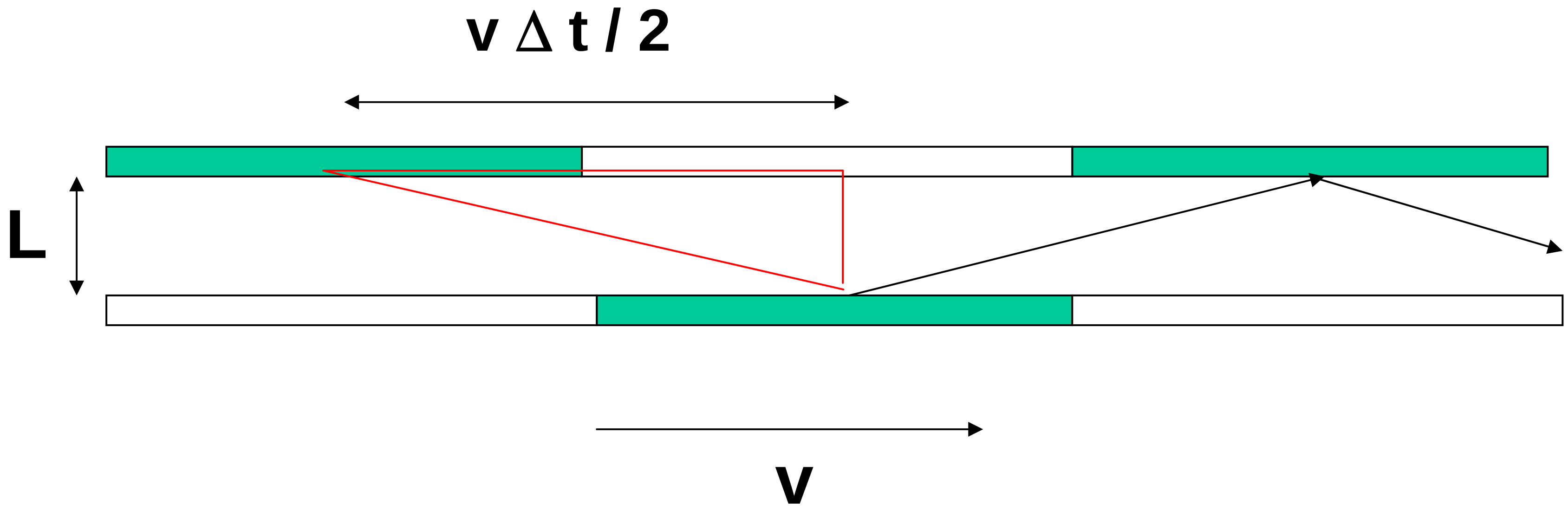
$$\Delta t_0 = \frac{2L}{c}$$

The moving light-clock



$$L^2 + \left(\frac{v\Delta t}{2}\right)^2 = \left(\frac{c\Delta t}{2}\right)^2$$

The moving light-clock



same value for c as for the clock at rest

$$L^2 + \left(\frac{v\Delta t}{2}\right)^2 = \left(\frac{c\Delta t}{2}\right)^2$$

We get:

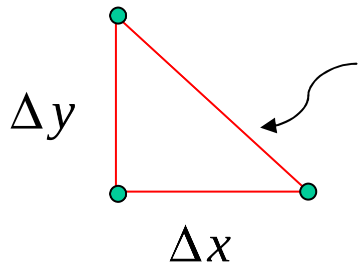
$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

a moving clock appears to be slowed down

It turns out that the concept of a ST metric tensor is of greatest value

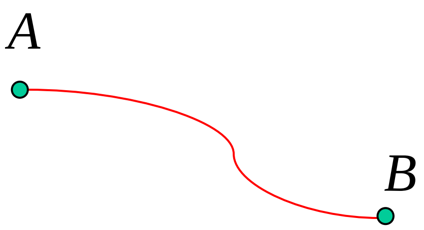
GRT: metric as fundamental object

• Pythagorean theorem in 2-dimensional Euclidean space



$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

• length of a curve



$$\ell = \int_A^B ds$$

$$ds^2 = dx^2 + dy^2$$

$$x = r\cos\theta; \quad y = r\sin\theta$$

$$dx = dr\cos\theta - r\sin\theta d\theta$$
$$dy = dr\sin\theta + r\cos\theta d\theta$$

$$dx^2 = \cos^2\theta dr^2 - 2r\sin\theta\cos\theta drd\theta + r^2\sin^2\theta d\theta^2$$
$$dy^2 = \sin^2\theta dr^2 + 2r\sin\theta\cos\theta drd\theta + r^2\cos^2\theta d\theta^2$$

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\, dx^i dx^j$$

Metric tensor: special relativity

• special relativity, inertial coordinates

$$x^{\mu} \equiv (x^0, x^i) = (ct, x, y, z)$$

• The constancy of the velocity of light in inertial coordinates

$$d\mathbf{x}^2 = c^2 dt^2$$

can be expressed as $ds^2 = 0$ where $ds^2 = -c^2 dt^2 + d\mathbf{x}^2$

$$\boxed{\begin{aligned} g_{00} &= -1, \\ g_{0i} &= 0, \\ g_{ij} &= \delta_{ij} = \mathrm{diag}(1,1,1). \end{aligned}}$$

Light rays are
null geodesics

Moreover, from the metric one immediately gets the observed time interval by

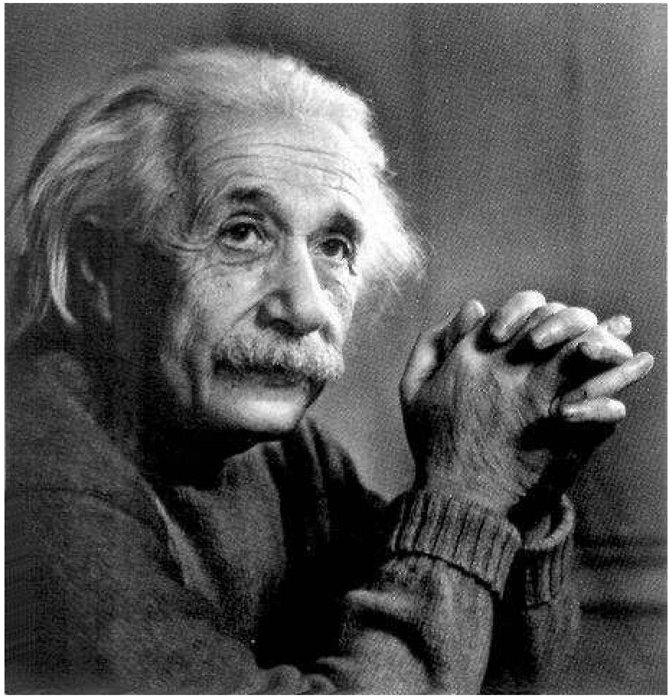
$$ds^2 = -c^2 d\tau^2 = -c^2 dt_0^2 = -c^2(dt^2 - d\mathbf{x}^2/c^2)$$

or

$$dt_0^2 = dt^2(1 - \mathbf{v}^2/c^2)$$

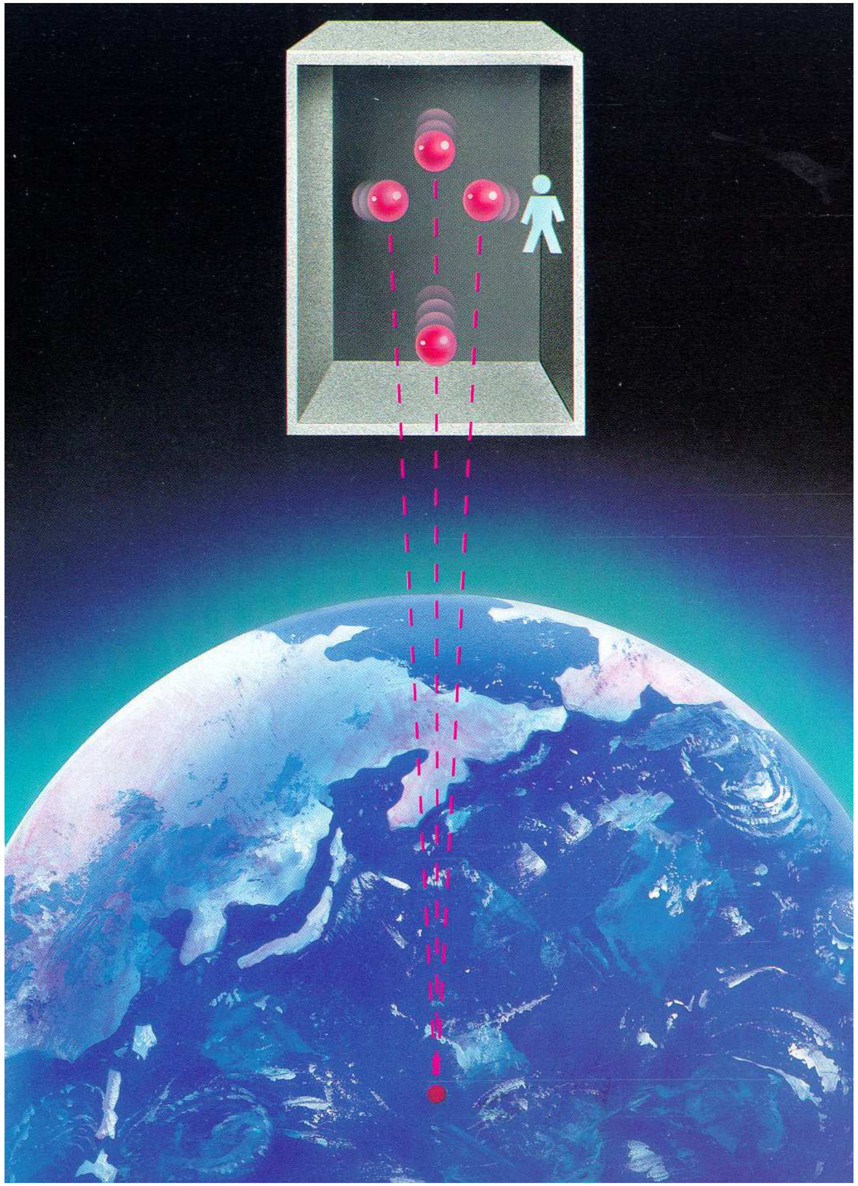
coordinate time interval

that agrees with the formula above



The gravitational field can also be described with the ST metric tensor

The reason is the **Equivalence Principle** (EP)

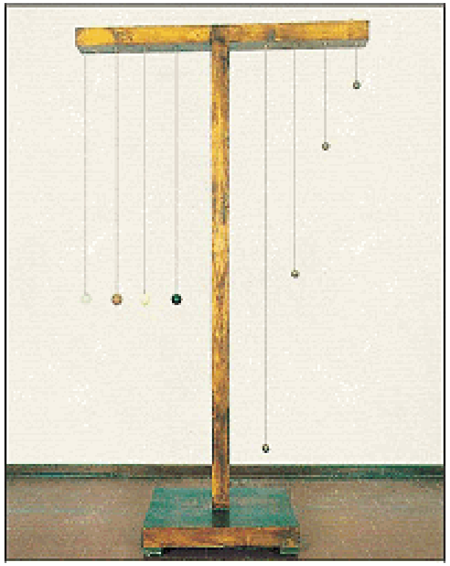


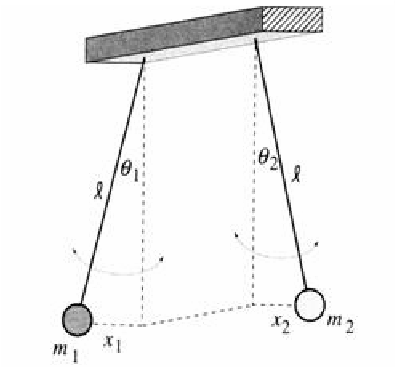
WEP:

Apart from tidal forces all uncharged test bodies fall at the same rate

inertial mass = heavy mass

WEP: pendelum measurements

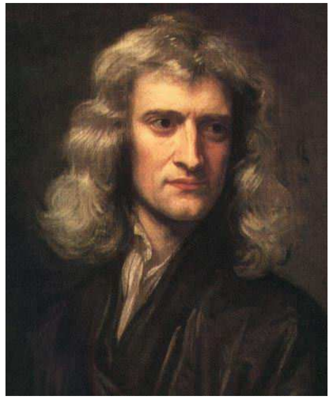




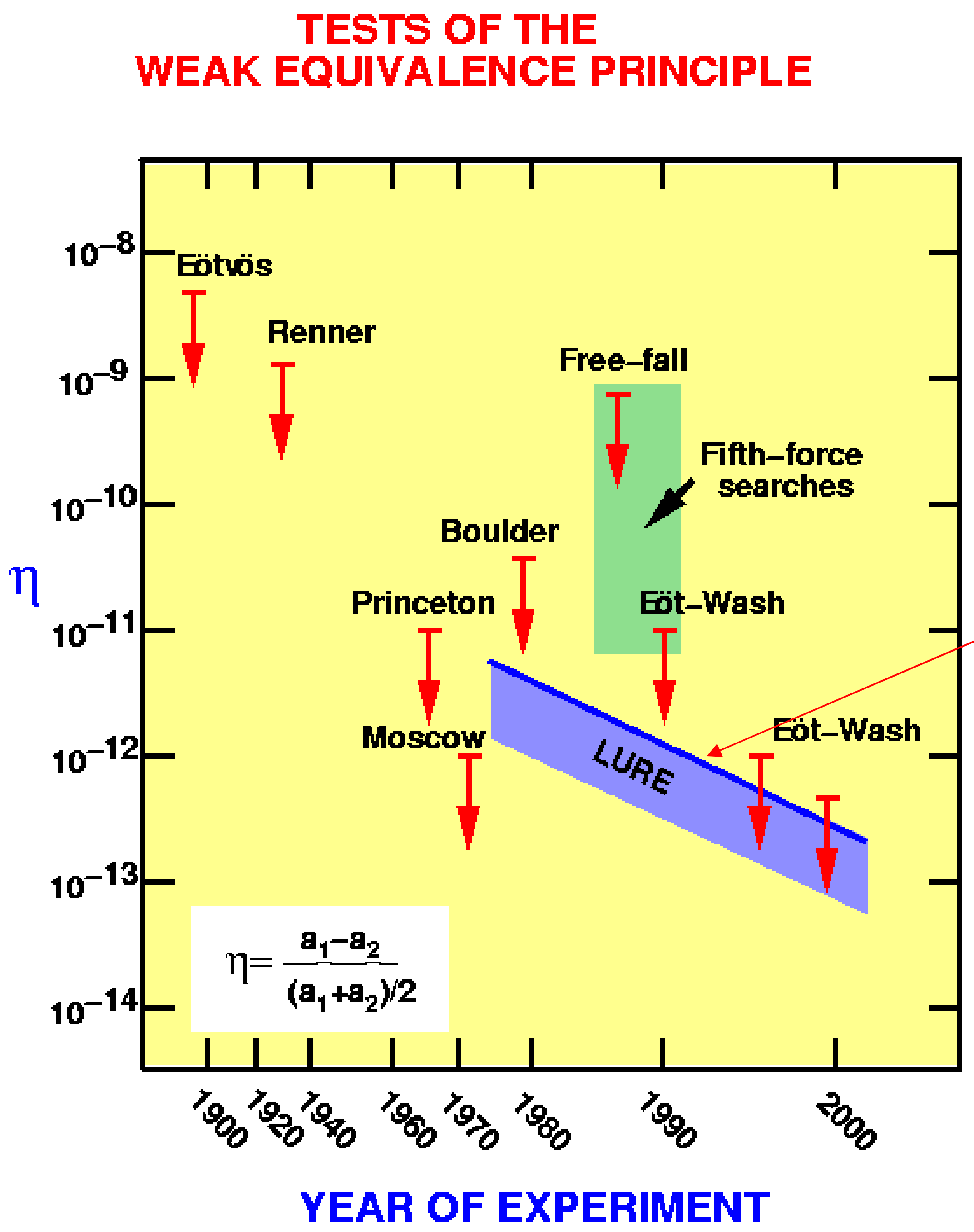
Different materials –
same swinging periods (l = const.)

Galileo Galilei	(1590-1638)	0.02
Isaac Newton	(1680)	0.001
Friedrich Bessel	(1830)	0.000017

relative accuracy



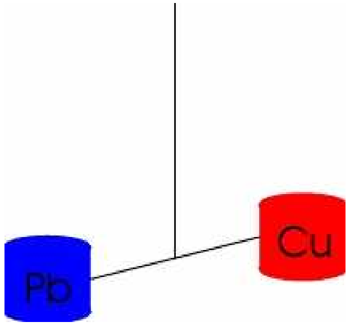
Friedrich Wilhelm Bessel
(1784-1846)

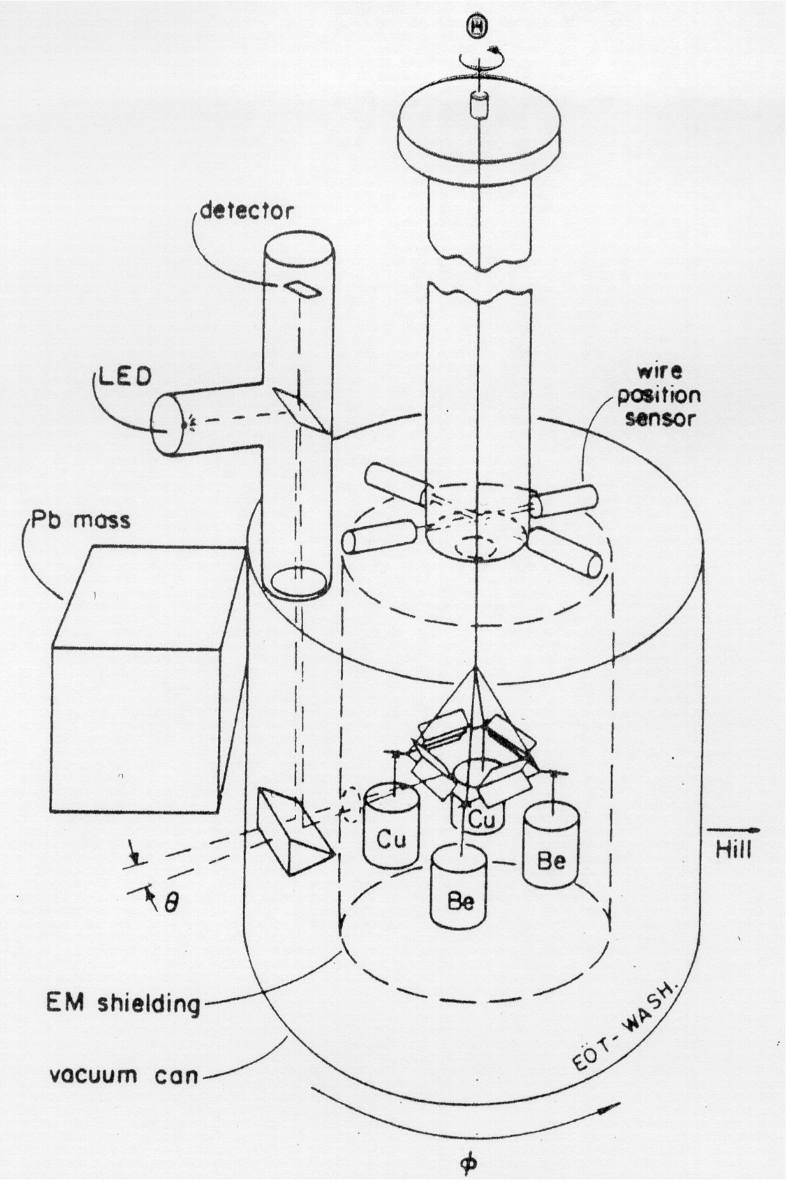


Earth and Moon in free-fall towards Sun (LLR)

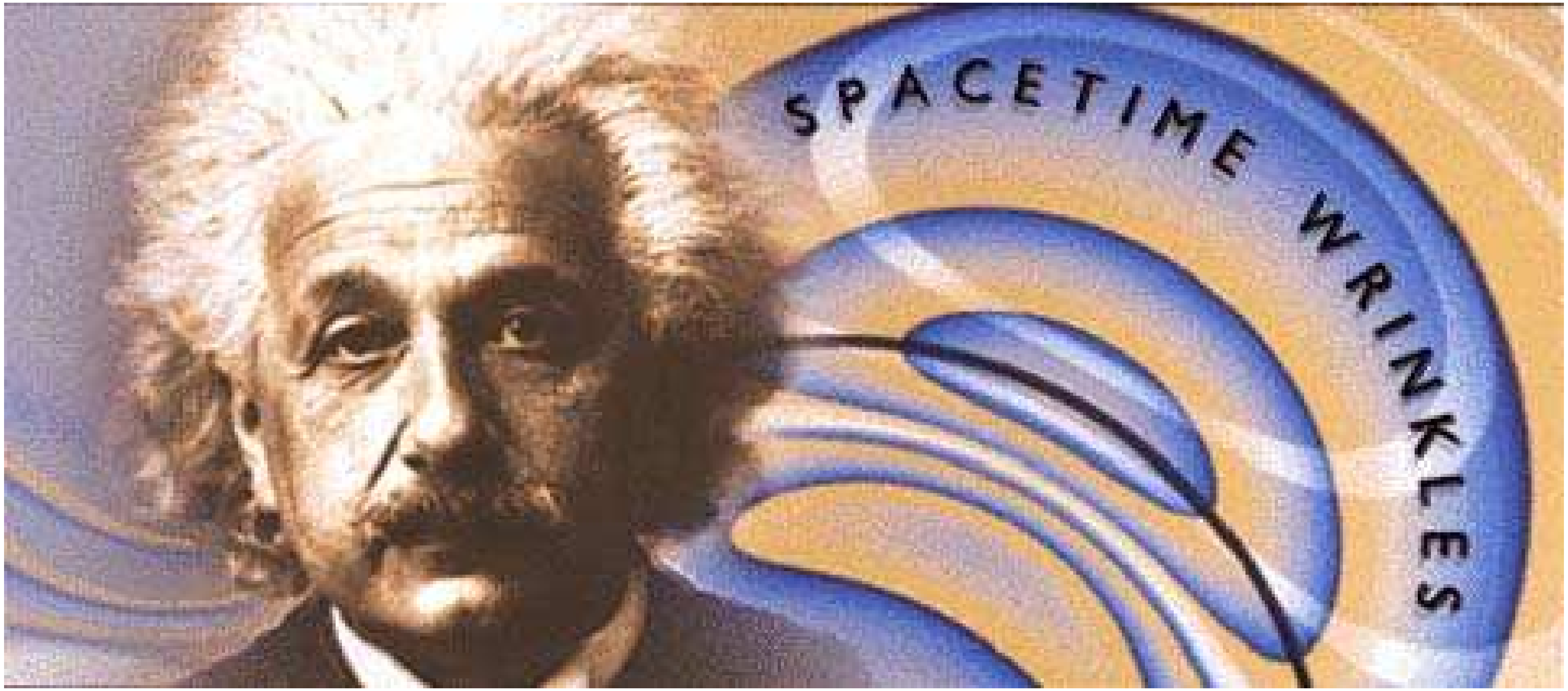
WEP: Torsion pendulums

Loránd Eötvös (1848-1919)



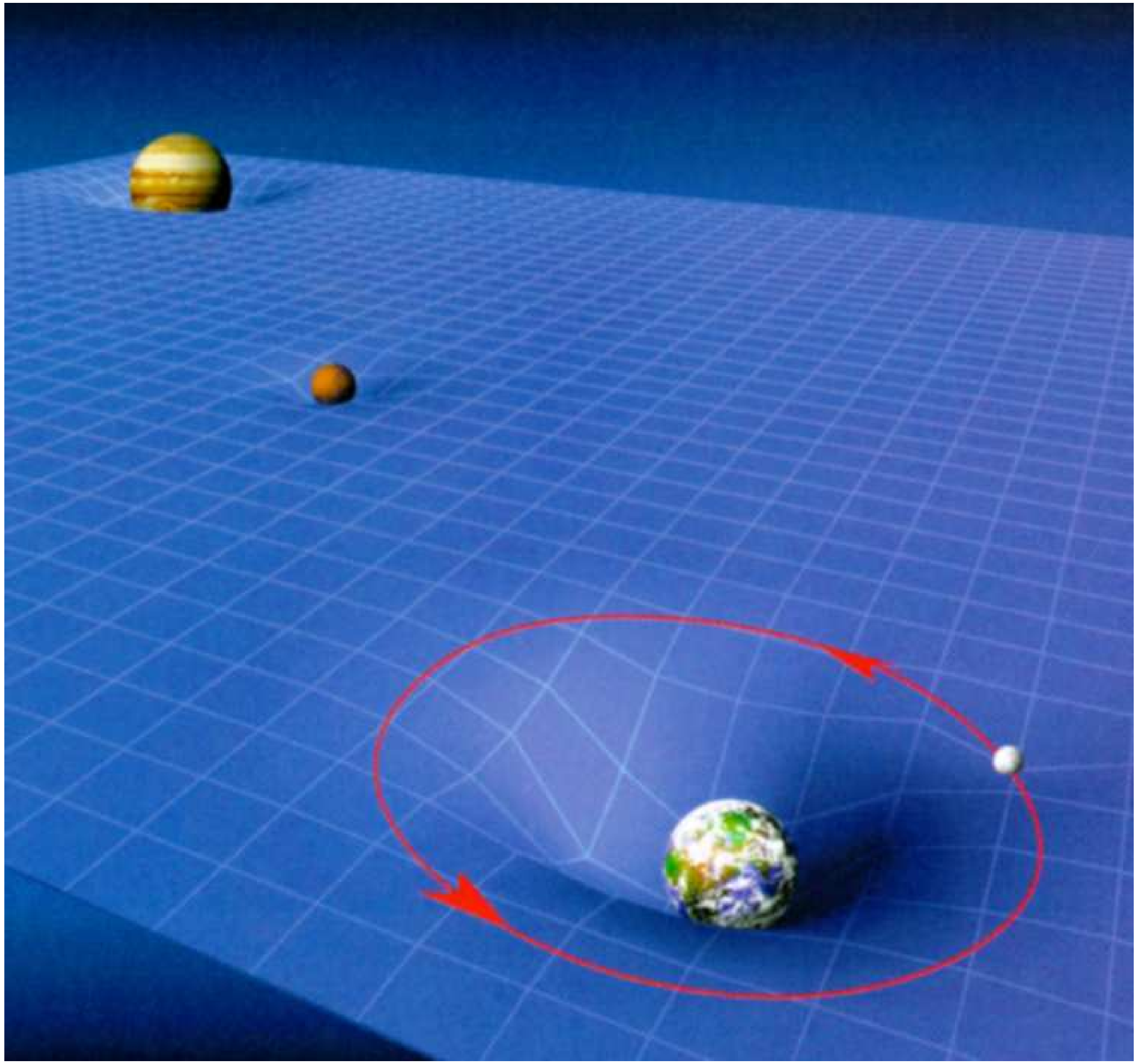


Eötvös	(1909)	5×10^{-9}
Braginsky-Panov	(1972)	10^{-12}
Adelberger	(2003)	5×10^{-13}



A.Einstein:

Gravity can be understood as effect of space-time curvature



Gravity
as phenomenon
of space-time
curvature

Precondition:
Equivalence Principle

Einstein's theory of gravity

To lowest order the gravitational potential U enters the metric tensor

$$
\begin{aligned}
g_{00} &= -1 + \frac{2U}{c^2} + O(c^{-4}) \\
g_{0i} &= O(c^{-3}) \\
g_{ij} &= \delta_{ij} + O(c^{-2})
\end{aligned}
\qquad (*)
$$

The Newtonian field equation (Poisson equation)

$$\Delta U = -4\pi G \rho$$

is contained in Einstein's field equations

$$\Phi(g_{\mu\nu}, \partial g_{\mu\nu}) = -4\pi G \mathcal{F}$$ matter variables

Einstein‘s field equations determine the metric tensor
up to four degrees of freedom that fix the coordinate system
(gauge freedom)

In the following only the so-called

harmonic gauge

will be used (generalized Cartesian inertial coordinates)

A ST reference system if determined

by

- Origin and spatial orientation of spatial coordinates

- Form of the metric tensor

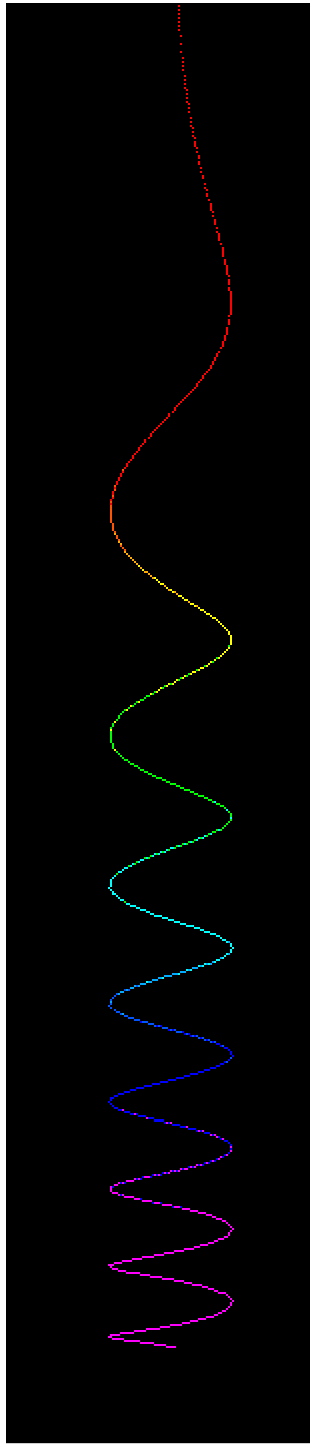
One consequence of U in the metric: gravitational redshift:

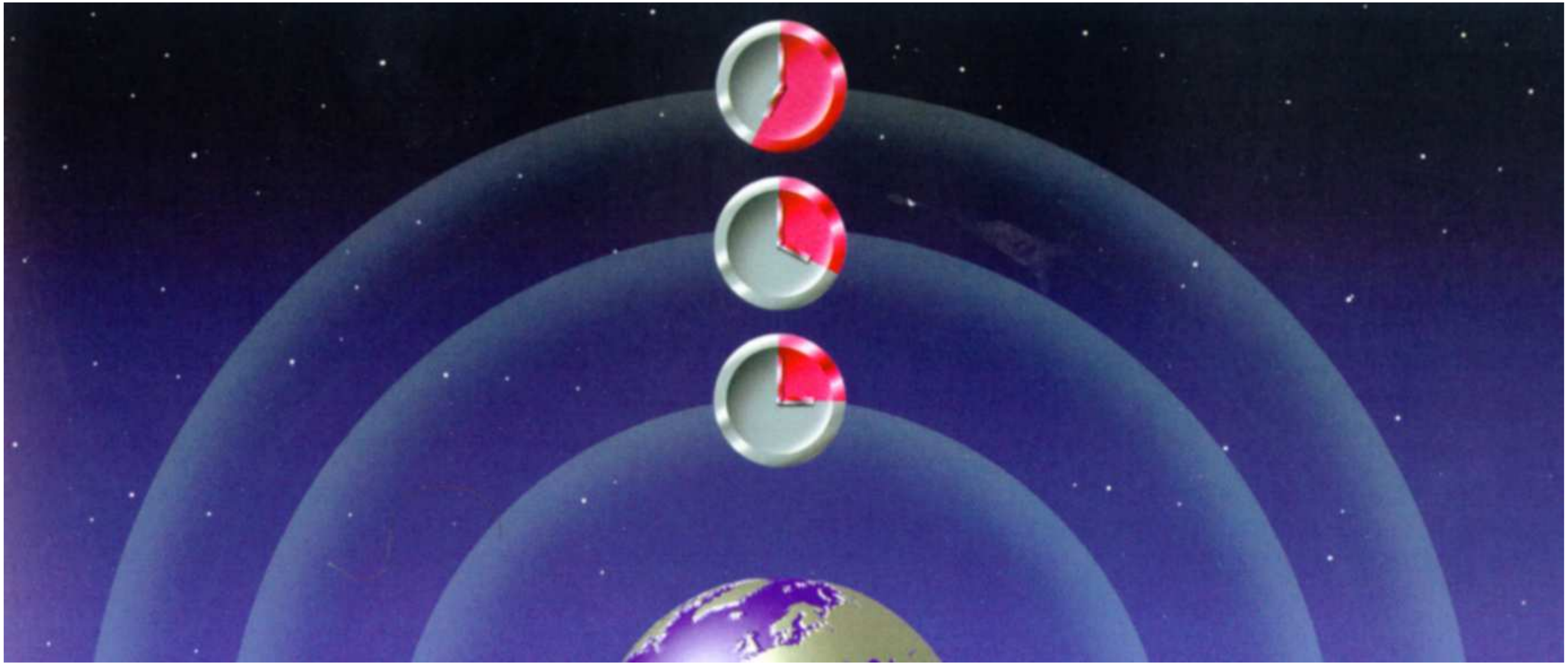
If a light signal propagates in a gravitational field from below to above its frequency appears to be reduced,

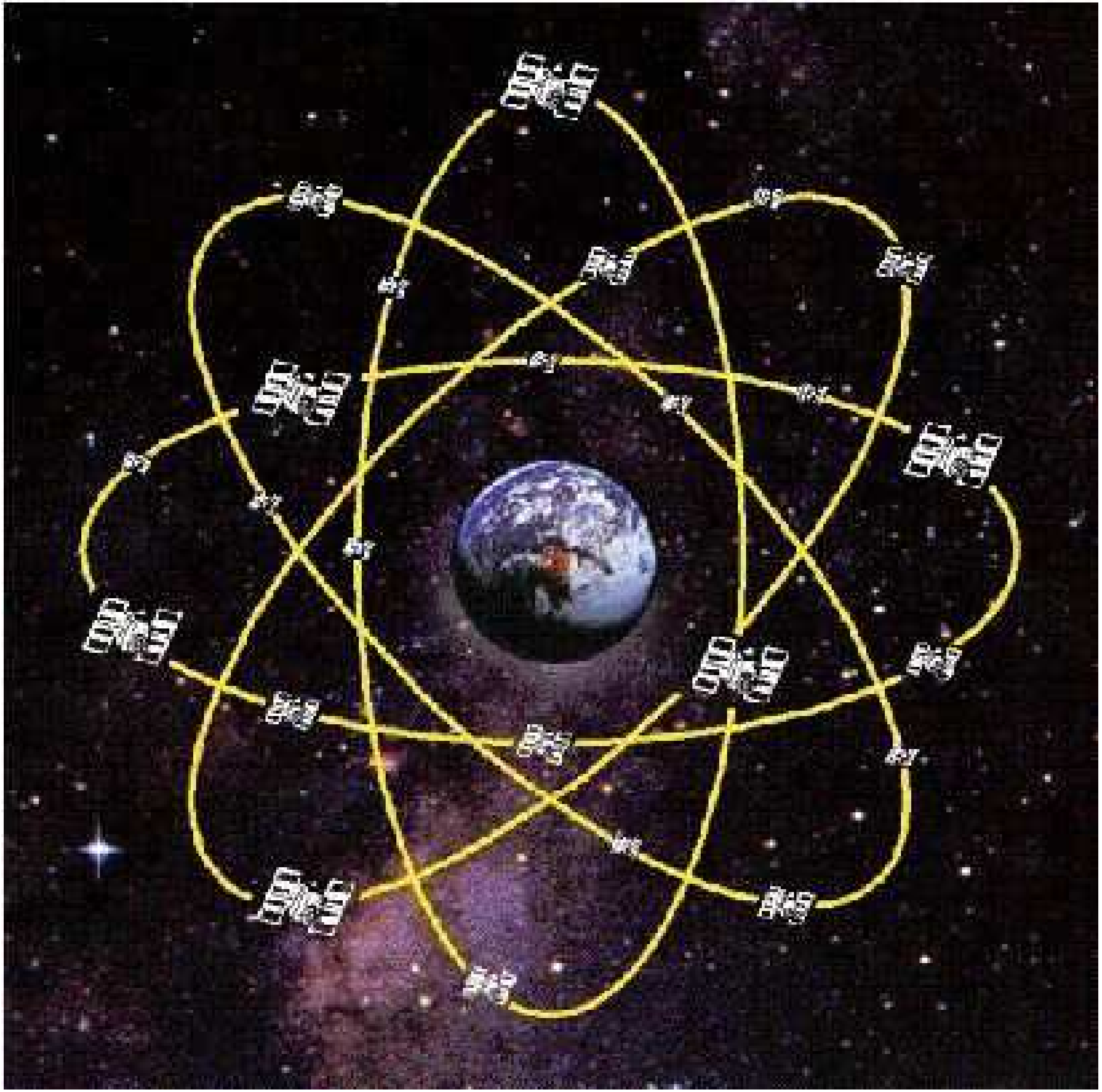
i.e., redshifted

Since the SI second is defined by the duration of a certain number of oscillations of a certain radiation resulting from Cs-133 atoms

$\rightarrow$ the rate of a clock depends upon its location in the gravity field







GPS:

24 satellites in 20 000 km height

emitting time signals

GPS accuracies

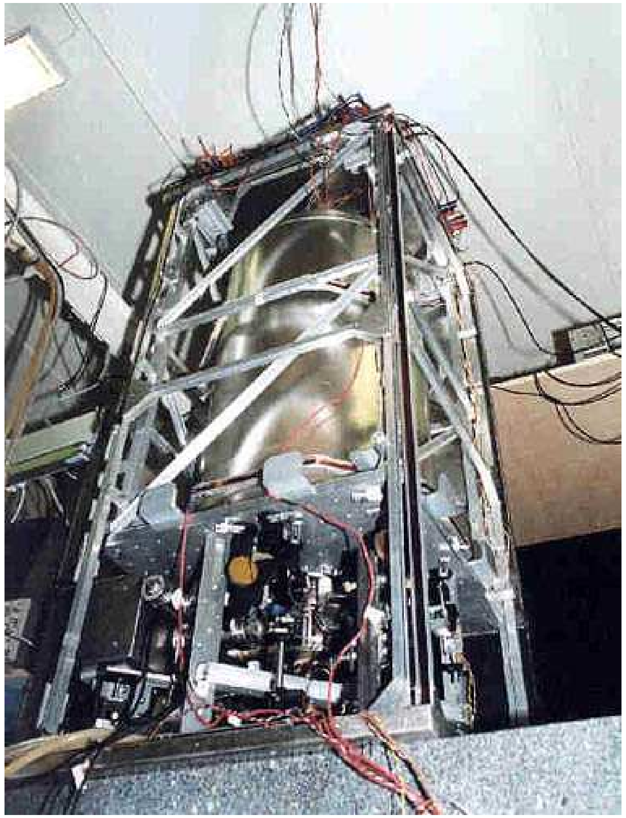
Positions: about 30 m

DGPS: cm – mm

at highest accuracies the action of gravity has to be taken into account

In the near future:
atomic clocks might be employed as gravimeters

Laser-cooled Cs - fountain clocks:
$\Delta f / f \approx 10^{-15}$



NIST Ytterbium optical clock at 10µK in optical lattice

10^{-18} **!**

(Age of universe: 4×10^{17} s)

A Geocentric RS: first approximation

$$ds^2 \simeq G_{00} c^2 dT^2 + G_{ab} dX^a dX^b$$

(*)
$$\simeq -\left(1 - \frac{2U}{c^2}\right) c^2 dT^2 + (d\mathbf{X})^2$$
$$= -\left(1 - \frac{2U}{c^2} - \frac{\mathbf{V}^2}{c^2}\right) c^2 dT^2 = -c^2 d\tau^2$$

$T = \mathrm{TCG}$ (Geocentric Coordinate Time), τ (proper) time of real clock

For earthbound clocks:

$$\frac{d\tau}{dT} \simeq 1 - \frac{U_{\mathrm{geo}}}{c^2} \simeq 1 - \frac{U_0}{c^2} + \frac{gh}{c^2}; \qquad U_{\mathrm{geo}} = U + \frac{1}{2}\mathbf{V}_{\mathrm{rot}}^2$$

$$f_{\mathrm{PTB}} \simeq (1 + 1.8 \times 10^{-13})\, f_{\mathrm{NBS}} \to \Delta\tau \simeq 5.4\, \mu\mathrm{s/a}$$

The timescale TT: it should differ from TCG by a constant Rate. Original idea: this rate should agree with that of a clock on the geoid. However: geoid not known to sufficient precision.

$$TT = k_E T = k_E \mathrm{TCG}$$

$$k_E = 1 - 6.969290134 \times 10^{-10} \qquad \text{(defining constant)}$$

The timescale TAI: practical realization of TT

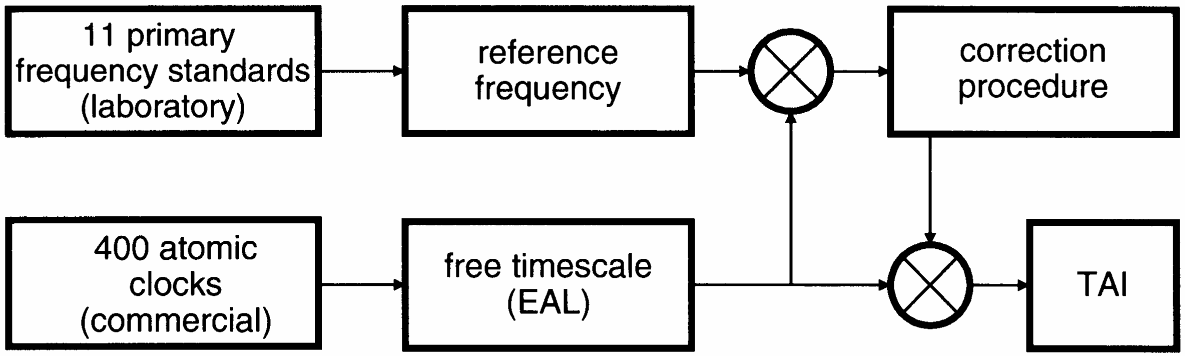


Fig. 2.24 Scheme how TAI is realized

TT, TAI and UTC

TT = TAI + 32.184 s

TAI = UTC + N s

leap seconds

ST reference systems with higher accuracy

Usually for applications within our solar system the (first) post-Newtonian approximation to Einstein‘s theory of gravity (in harmonic coordinates) is employed

The post-Newtonian framework

Slow-motion, weak field approximation

$$\epsilon^2 \sim \left(\frac{v}{c}\right)^2 \sim \frac{GM}{c^2 R} < 10^{-5} \qquad \text{in solar system}$$

$\longrightarrow$

$$\mathrm{EOM} = \mathrm{EOM}_{\mathrm{Newton}} + \epsilon^2 (\mathrm{EOM})_{\mathrm{1PN}} + \dots$$

Short history:

(1915 Einstein)

1916 Droste, De Sitter

1917 Lorentz, Droste

1937 Levi-Civita

1938 Chazy; Einstein, Infeld, Hoffmann

1939 Fock

1951 Papapetrou

1965 Chandrasekhar

1981 Caporali

1985 Grishuk, Kopejkin

1989 Brumberg, Kopejkin

1991 Damour, Soffel, Xu

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General-relativistic celestial mechanics. I. Method and definition of reference systems

Thibault Damour
Institut des Hautes Etudes Scientifiques, 91440 Bures sur Yvette, France
and Département d'Astrophysique Relativiste et de Cosmologie, Observatoire de Paris,
Centre National de la Recherche Scientifique, 92195 Meudon CEDEX, France

Michael Soffel and Chongming Xu*
Theoretische Astrophysik, Universität Tübingen, Auf der Morgenstelle 10, 7400 Tübingen,
Federal Republic of Germany
(Received 2 November 1990)

We present a new formalism for treating the general-relativistic celestial mechanics of systems of N arbitrarily composed and shaped, weakly self-gravitating, rotating, deformable bodies. This formalism is aimed at yielding a complete description, at the first post-Newtonian approximation level, of (i) the global dynamics of such N-body systems ("external problem"), (ii) the local gravitational structure of each body ("internal problem"), and, (iii) the way the external and the internal problems fit together ("theory of reference systems"). This formalism uses in a complementary manner $N+1$ coordinate charts (or "reference systems"): one "global" chart for describing the overall dynamics of the N bodies, and N "local" charts adapted to the separate description of the structure and environment of each body. The main tool which allows us to develop, in an elegant manner, a constructive theory of these $N+1$ reference systems is a systematic use of a particular "exponential" parametrization of the metric tensor which has the effect of linearizing both the field equations, and the transformation laws under a change of reference system. This linearity allows a treatment of the first post-Newtonian relativistic celestial mechanics which is, from a structural point of view, nearly as simple and transparent as its Newtonian analogue. Our scheme differs from previous attempts in several other respects: the structure of the stress-energy tensor is left completely open; the spatial coordinate grid (in each system) is fixed by algebraic conditions while a convenient "gauge" flexibility is left open in the time coordinate [at the order $\delta t = O(c^{-4})$]; the gravitational field locally generated by each body is skeletonized by particular relativistic multipole moments recently introduced by Blanchet and Damour, while the external gravitational field experienced by each body is expanded in terms of a particular new set of relativistic tidal moments. In this first paper we lay the foundations of our formalism, with special emphasis on the definition and properties of the N local reference systems, and on the general structure and transformation properties of the gravitational field. As an illustration of our approach we treat in detail the simple case where each body can, in some approximation, be considered as generating a spherically symmetric gravitational field. This "monopole truncation" leads us to a new (and, in our opinion, improved) derivation of the Lorentz-Droste-Einstein-Infeld-Hoffmann equations of motion. The detailed treatment of the relativistic motion of bodies endowed with arbitrary multipole structure will be the subject of subsequent publications.

Canonical form of the PN harmonic metric

$$\boxed{\begin{aligned} g_{00} &= -1 + \frac{2}{c^2} w(t, \boldsymbol{x}) - \frac{2}{c^4} w^2(t, \boldsymbol{x}) \, , \\ g_{0i} &= -\frac{4}{c^3} w^i(t, \boldsymbol{x}) \, , \\ g_{ij} &= \delta_{ij} \left(1 + \frac{2}{c^2} w(t, \boldsymbol{x}) \right) . \end{aligned}}$$

w: gravito-electric potential, generalizes U

w^i: gravito-magnetic potential (Lense-Thirring effects)

Celestial RS: quasi-inertial, no-rotation w.r.t. remote Astronomical objects (quasars)

We have to distinguish a

BCRS (Barycentric Celestial Reference System)

from a

GCRS (Geocentric Celestial Reference Sytem)

For certain applications we need even more CRS

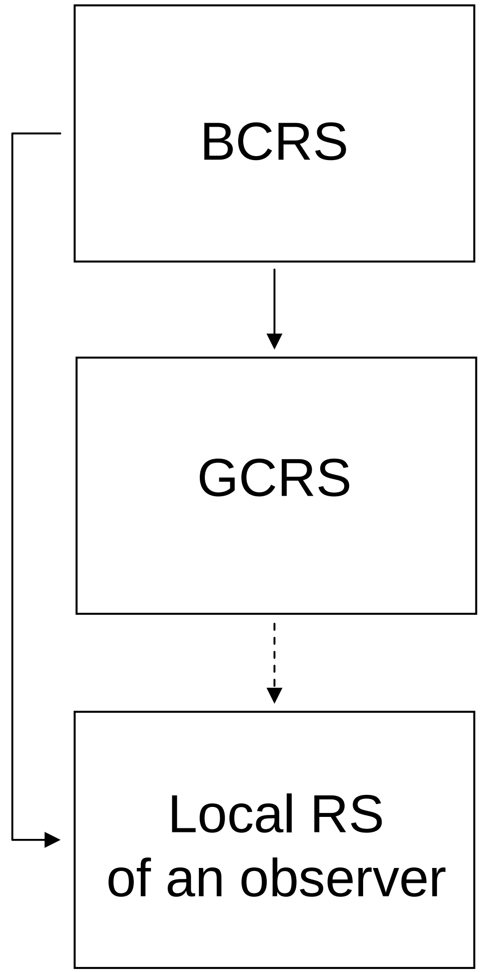
Metric tensor and reference systems

- In relativistic astronomy the
  - BCRS (Barycentric Celestial Reference System)
  - GCRS (Geocentric Celestial Reference System)
  - Local reference system of an observer

  play an important role.

- All these reference systems are defined by

  the form of the corresponding metric tensor.



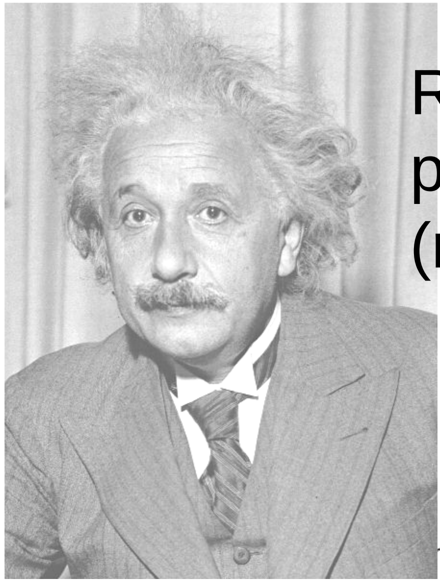
But: the RS are just coordinate systems that can be chosen in many ways (they have no physical meaning)

$\longrightarrow$

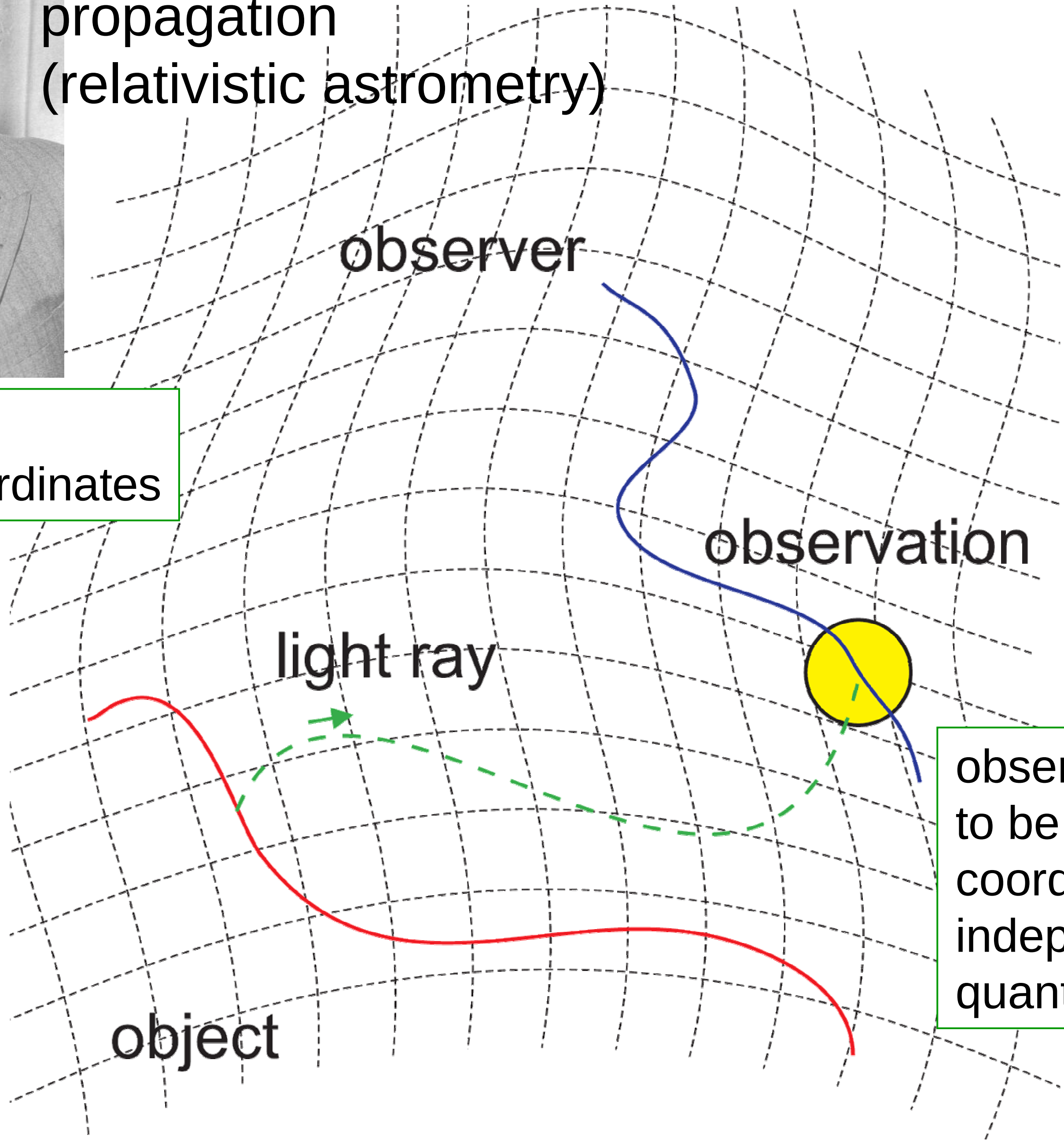
In addition to the RS we need theories for the

- observables
- associated techniques

- signal propagation



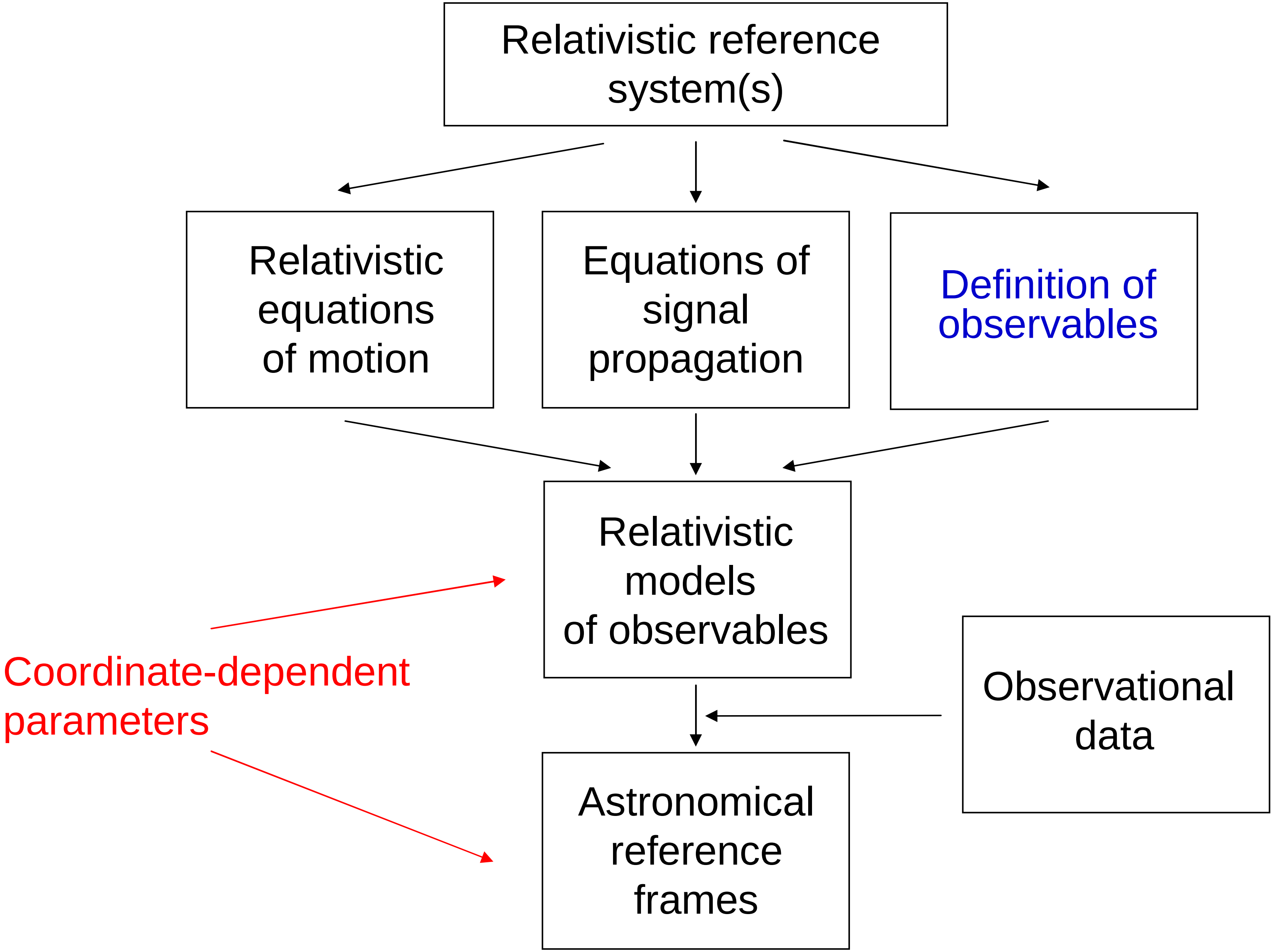
Relativistic theory of light propagation
(relativistic astrometry)



no physically preferred coordinates

observables have to be computed as coordinate independent quantities

Reference systems, frames and observables in GRT

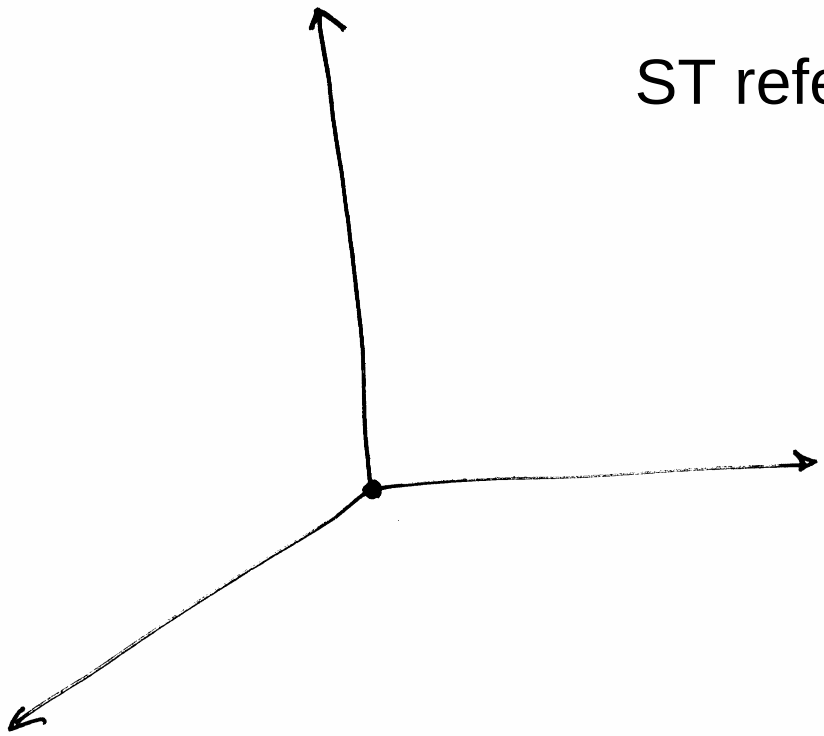


Relativistic theory of observables: examples

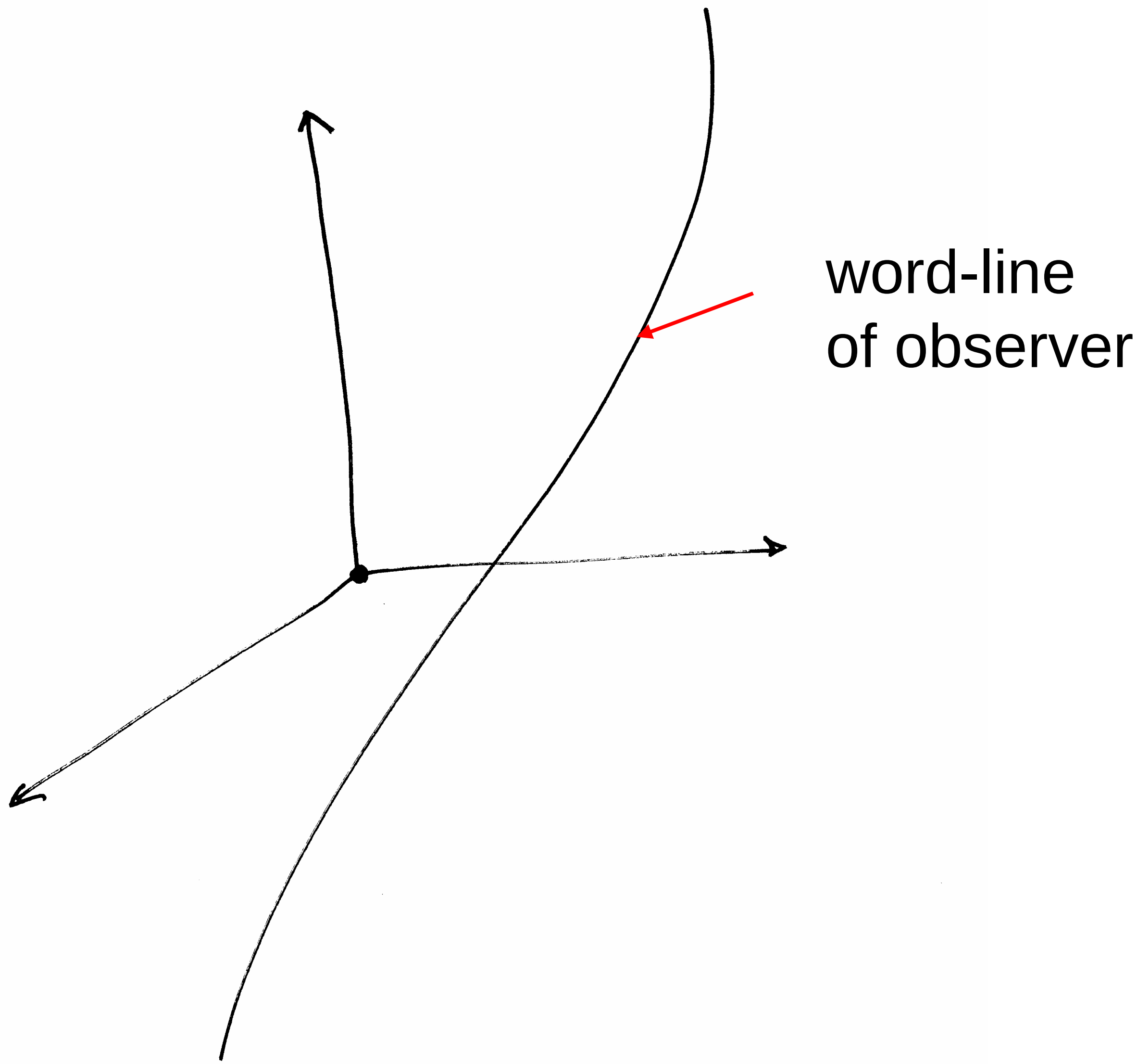
- proper time interval dτ: obtained directly from the metric
- (ds along the clock‘s worldline)

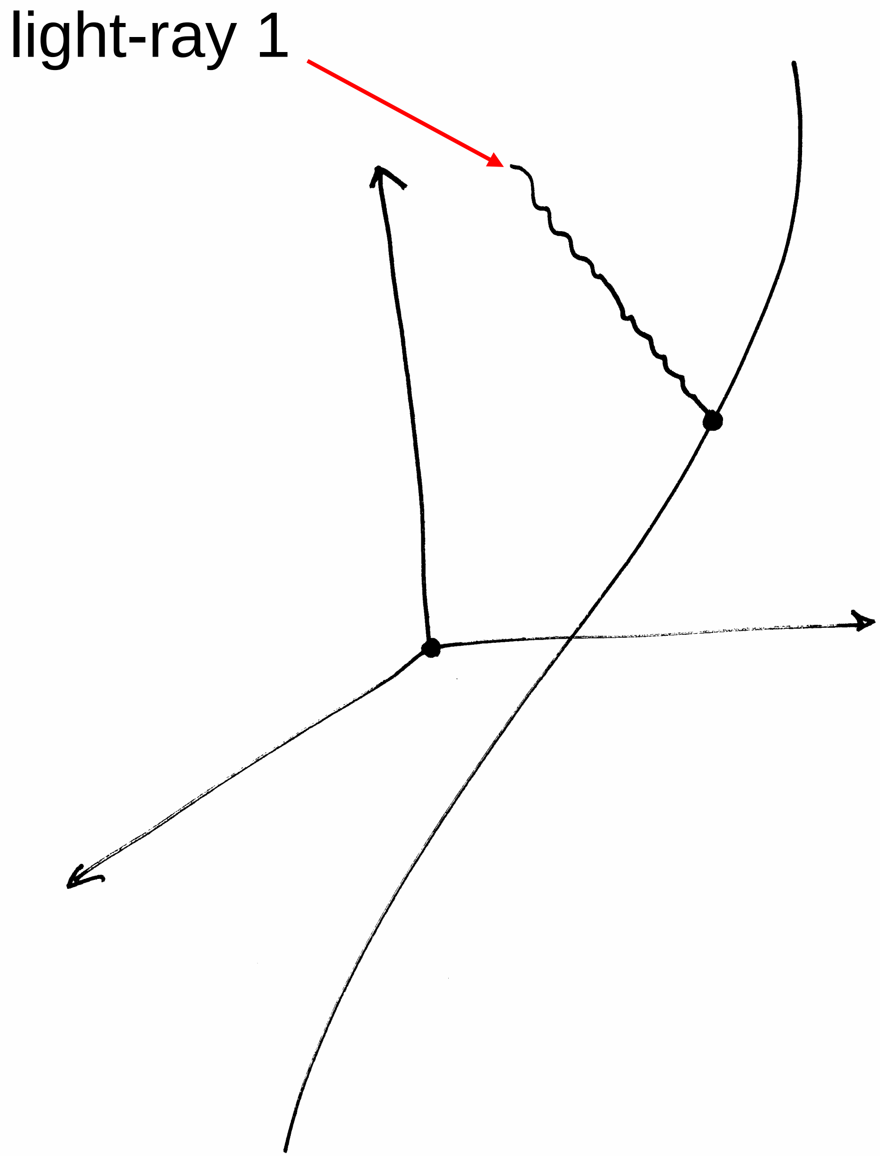
$$d\tau^2 = -\frac{1}{c^2} ds^2$$

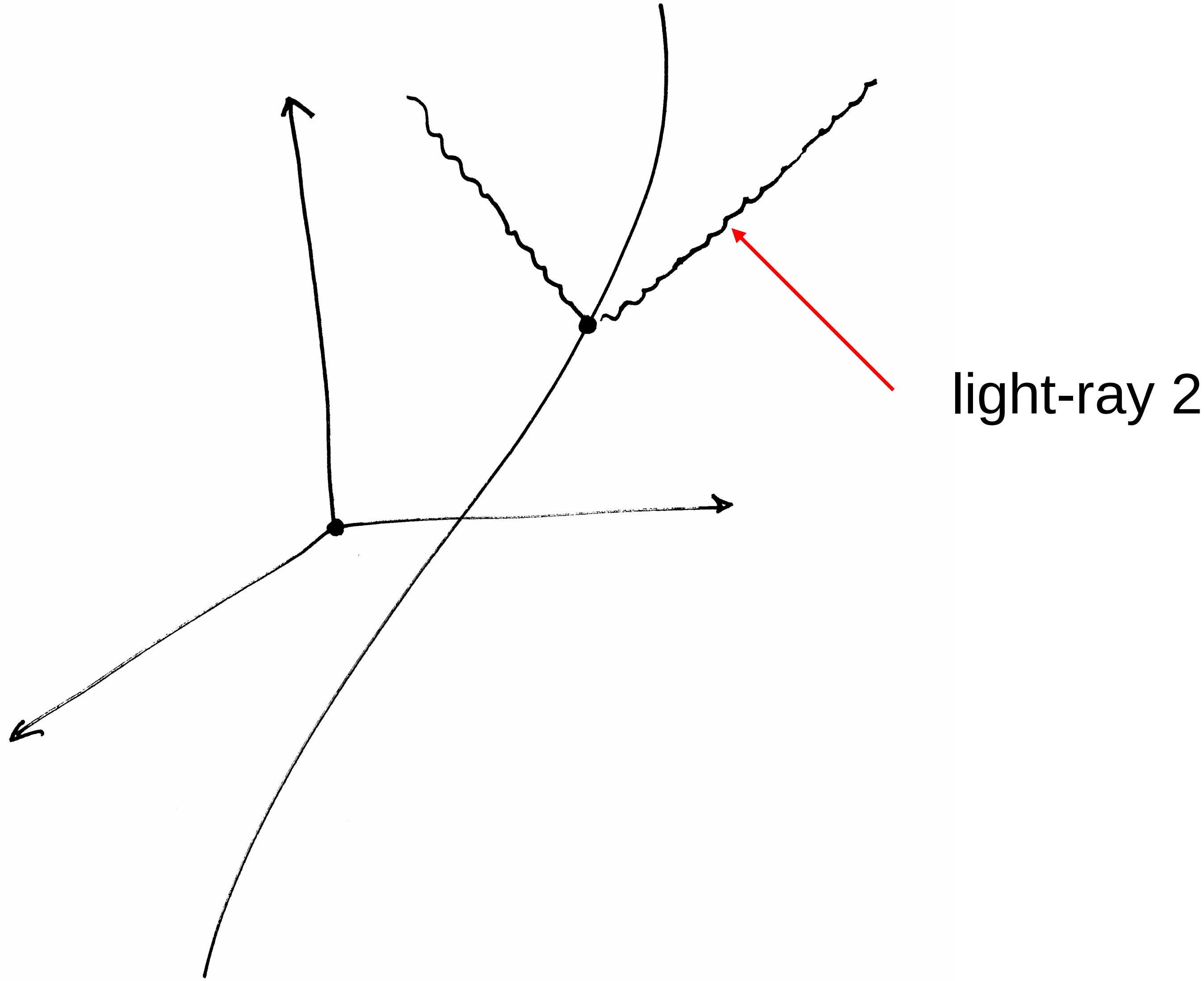
- observed angle between two incident light rays 1 and 2

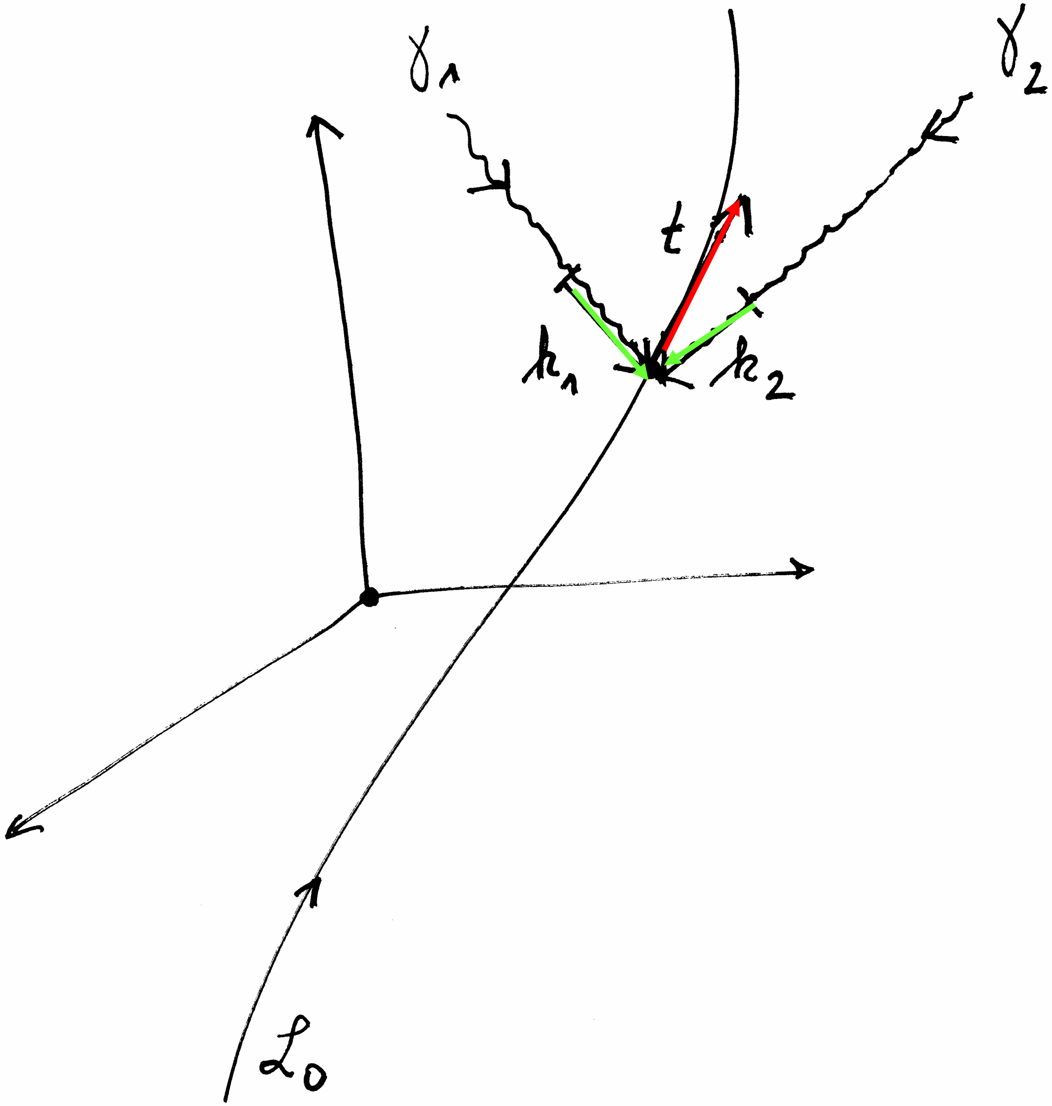


ST reference system


word-line
of observer


light-ray 1

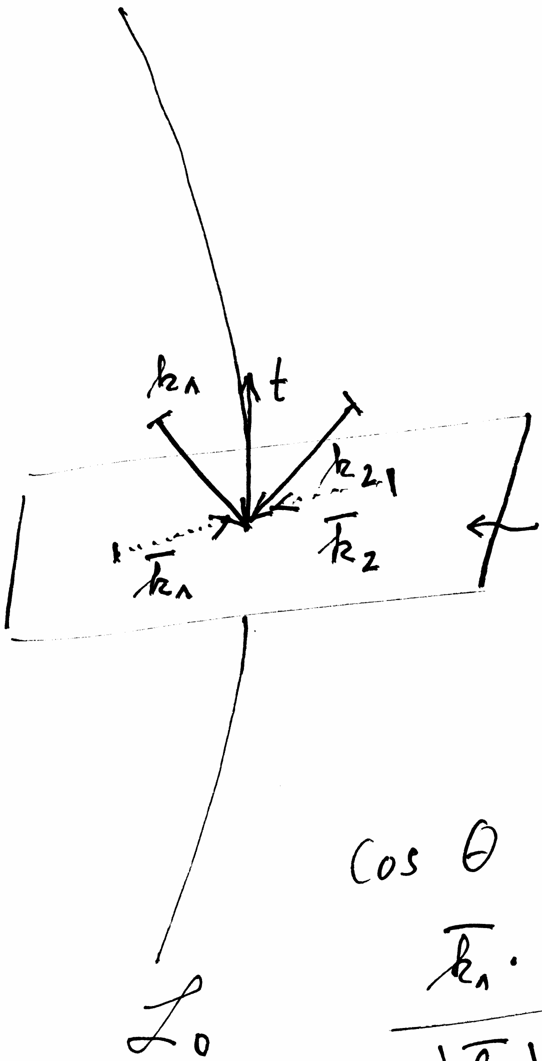

light-ray 2



3 ST tangent vectors:

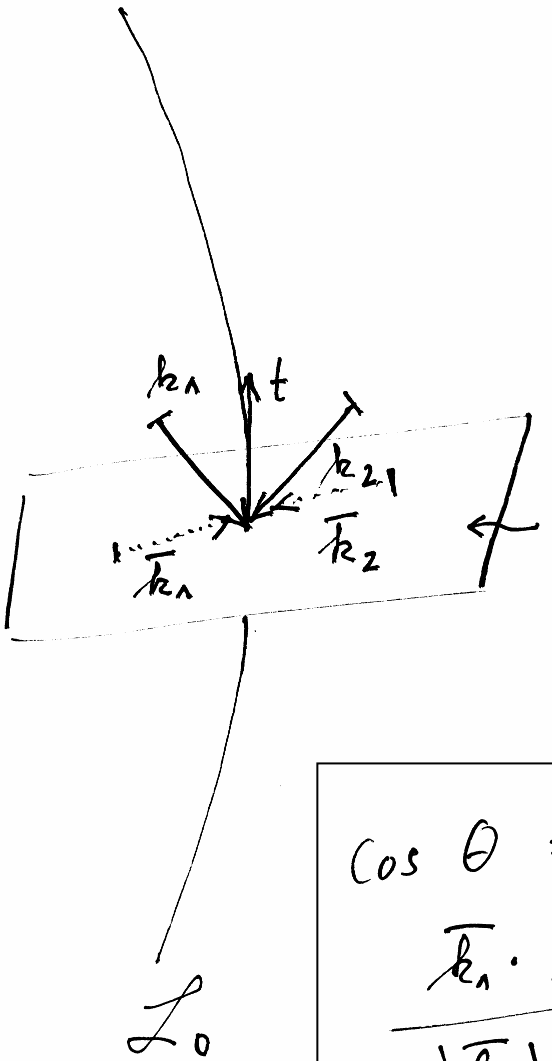
t to observer's world-line

k1 and **k2** to the two light-rays



3-space as experienced by observer

$$\cos\theta = \frac{\bar{k}_1 \cdot \bar{k}_2}{|\bar{k}_1|\,|\bar{k}_2|}$$

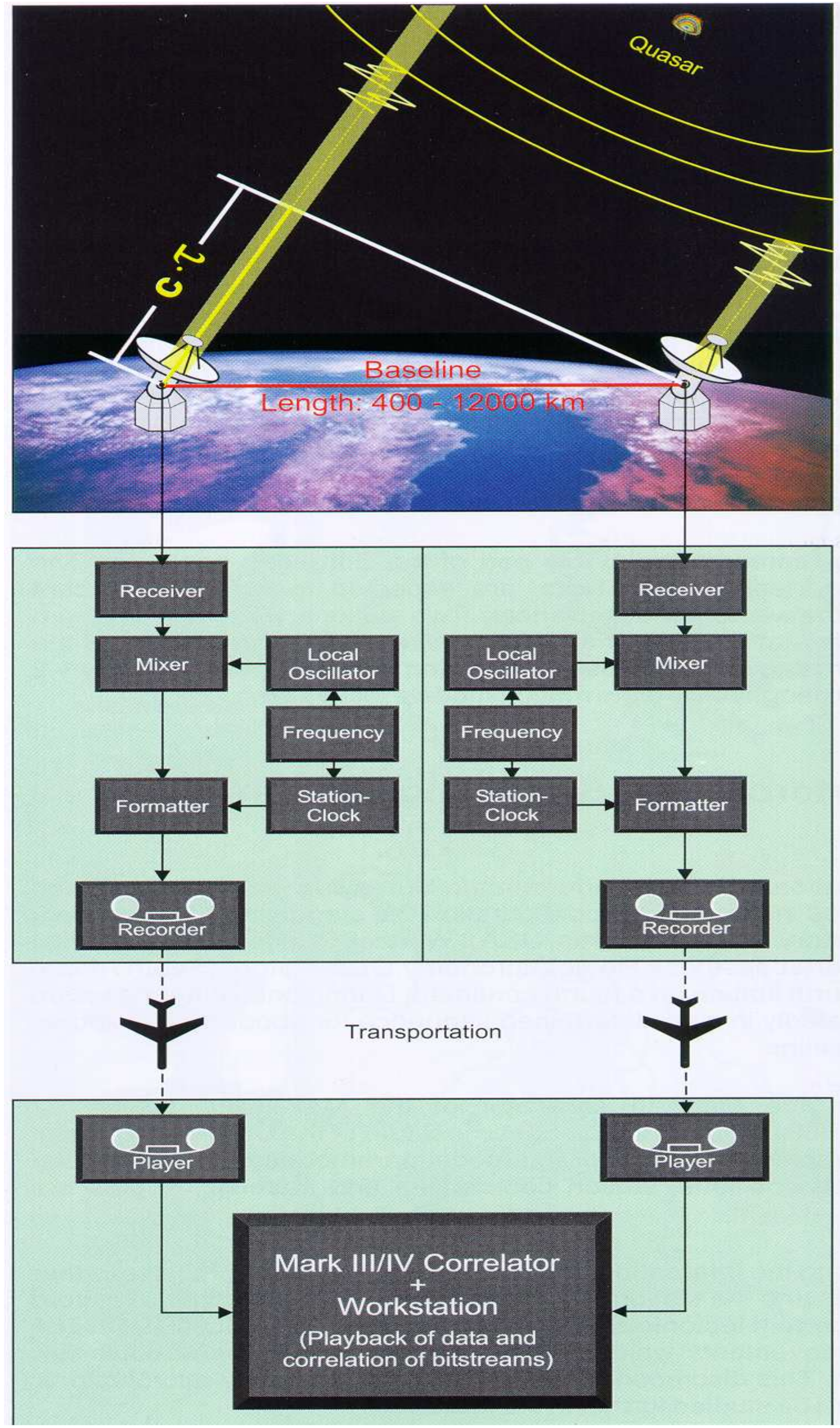


3-space as experienced by observer; $\bar{k}$: projection into 3-space

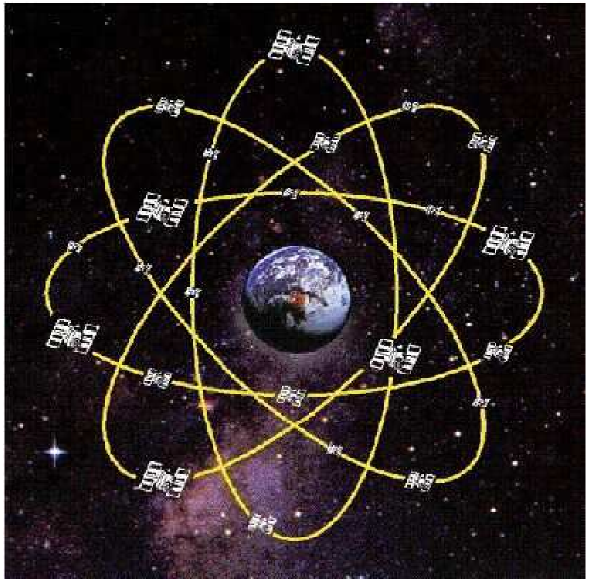
$$\boxed{\cos\theta = \frac{\bar{k}_1 \cdot \bar{k}_2}{|\bar{k}_1|\,|\bar{k}_2|}}$$

Projection, scalar-product and norm involve the metric tensor

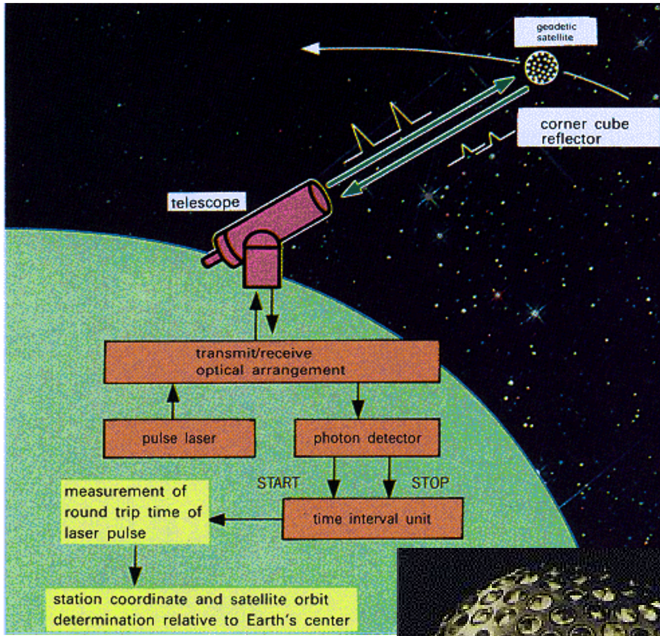
Relativistic metrology



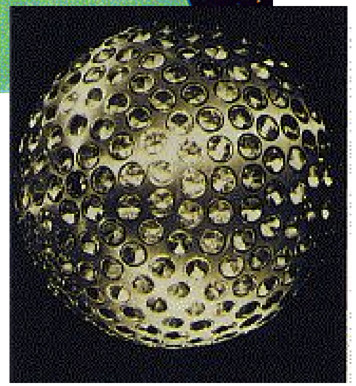
21-m VLBI antenna
Wettzell, Germany



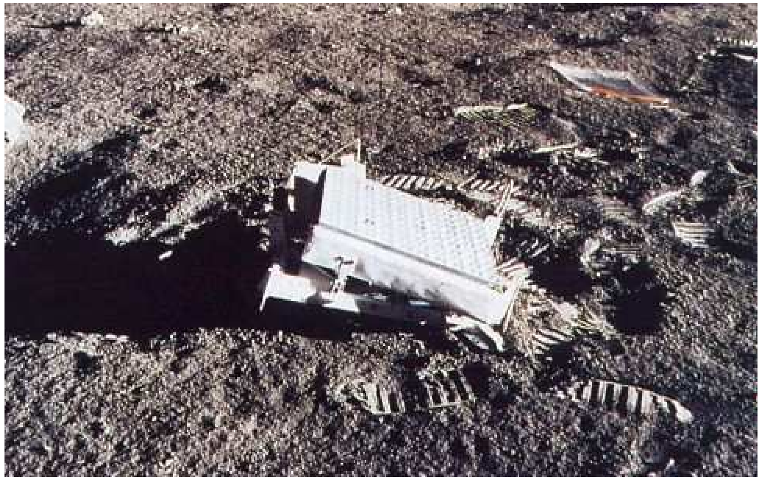
GPS



geodetic satellite
corner cube reflector
telescope
transmit/receive optical arrangement
pulse laser
photon detector
START
STOP
time interval unit
measurement of round trip time of laser pulse
station coordinate and satellite orbit determination relative to Earth's center

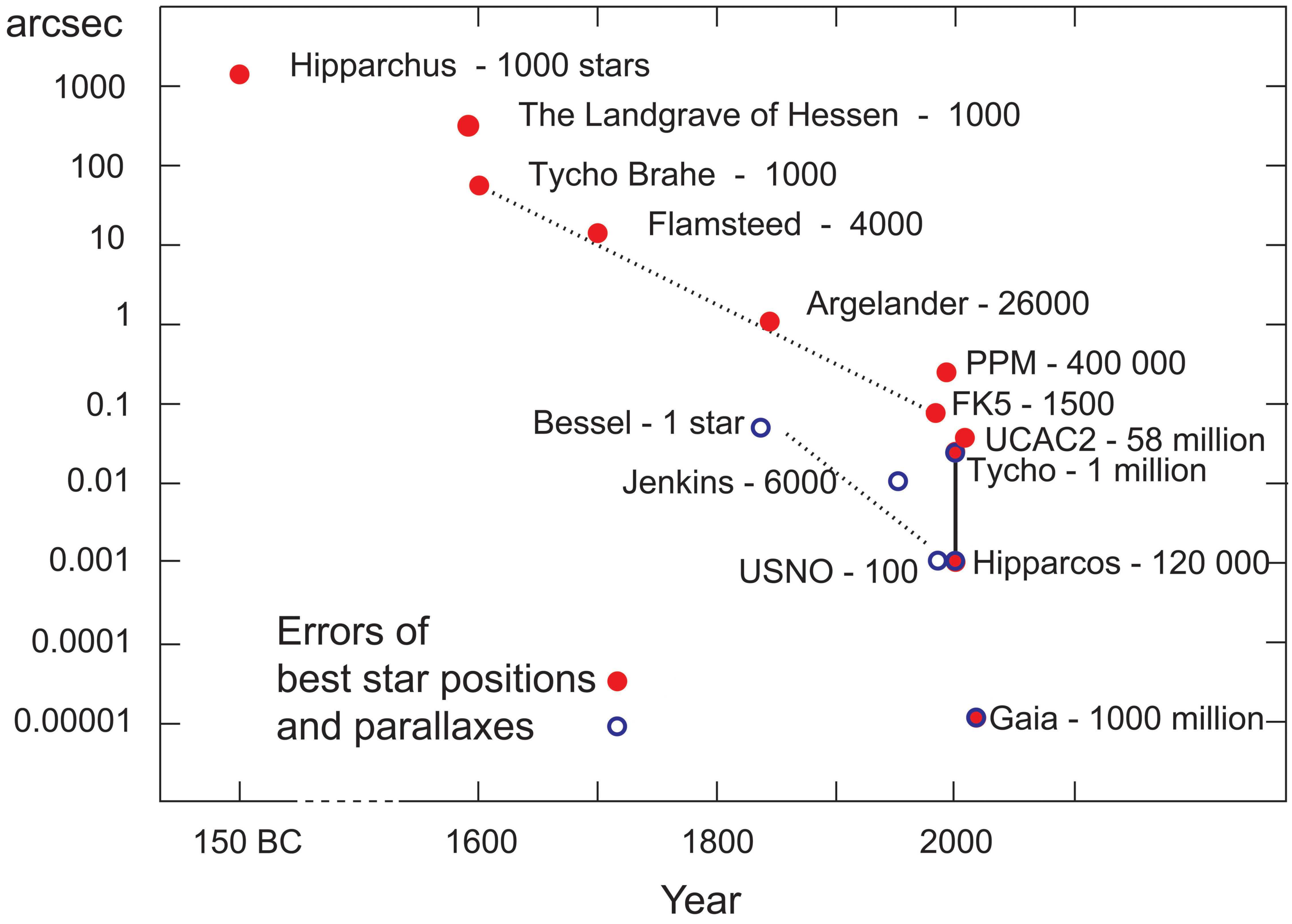


SLR

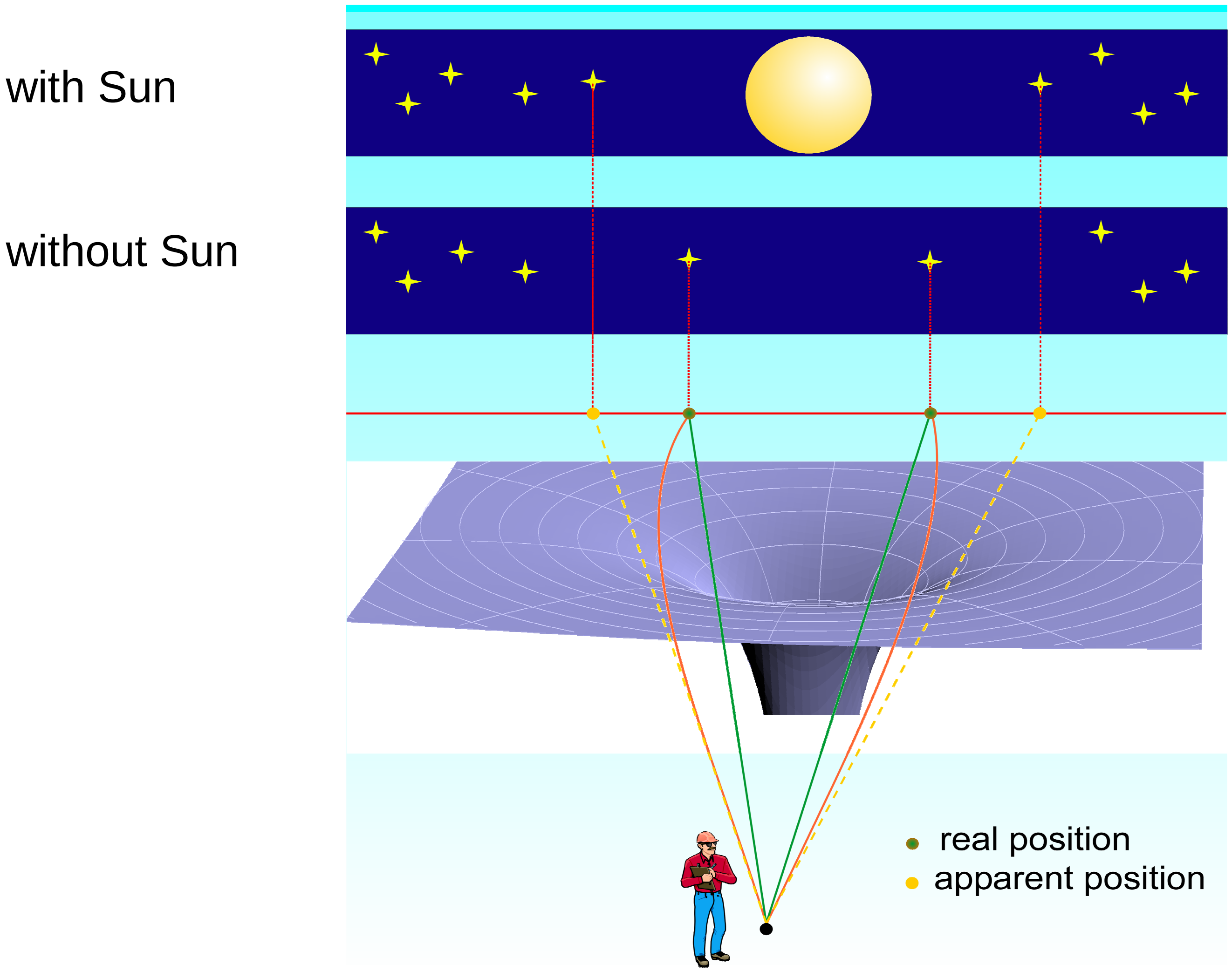


LLR

Astrometry: accuracies



Gravitational light deflection



equations of light propagation

- The equations of light propagation in the BCRS

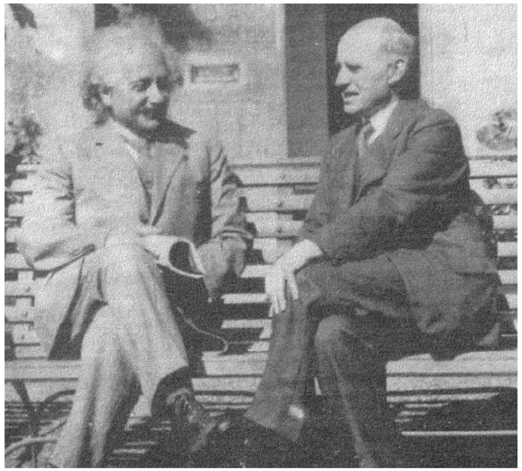
$$g_{00} = -1 + \frac{2}{c^2} w(t, \boldsymbol{x})$$

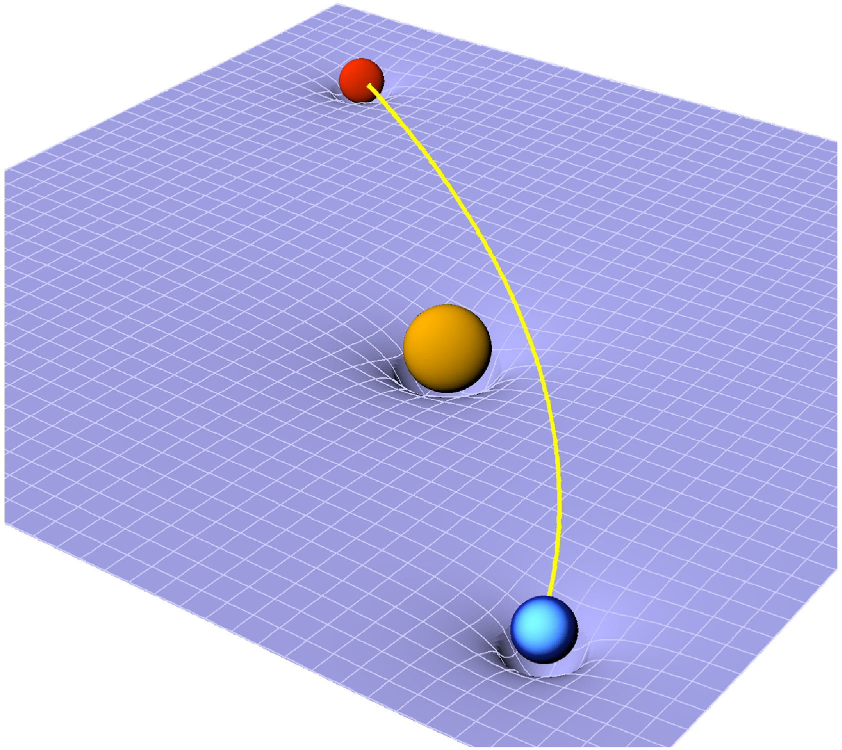
$$g_{0i} = 0$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2} w(t, \boldsymbol{x})\right)$$

- Relativistic corrections to the “Newtonian” straight line:

$$\boldsymbol{x}(t) = \boldsymbol{x}_0(t) + c\boldsymbol{\sigma}(t - t_0) + \frac{1}{c^2}\Delta\boldsymbol{x}(t)$$





Gravitational light deflection

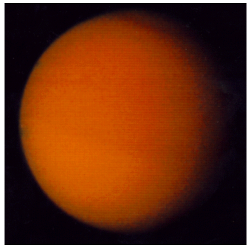
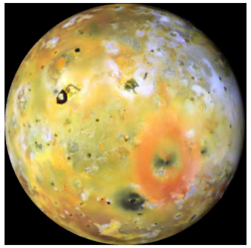
- The principal effects due to the major bodies of the solar system in μas
- The maximal angular distance to the bodies where the effect is still >1 μas

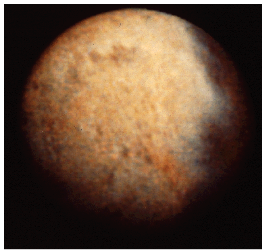
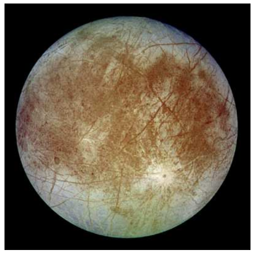
body	Monopole	ψ_{max}	Quadrupole	ψ_{max}	ppN	ψ_{max}
Sun	1.75×10^6	180 °			11	53′
(Mercury)	83	9 ′				
Venus	493	4.5 °				
Earth	574	125 °				
Moon	26	5 °				
Mars	116	25 ′				
Jupiter	16270	90 °	240	152 ″		
Saturn	5780	17 °	95	46 ″		
Uranus	2080	71 ′	8	4 ″		
Neptune	2533	51 ′	10	3 ″		

Gravitational light deflection: moons, minor planets

- A body of mean density ρ produces a light deflection exceeding δ if its radius:

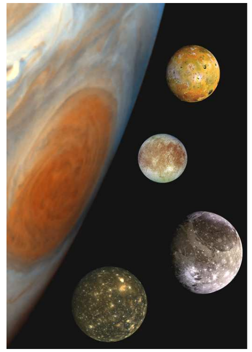
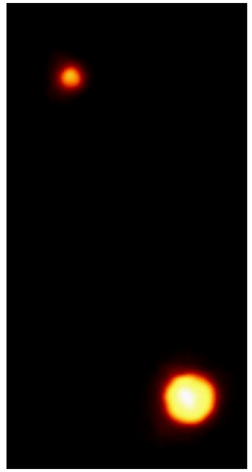
$$R \geq \left(\frac{\rho}{1\,\mathrm{g/cm}^3} \right)^{-1/2} \times \left(\frac{\delta}{1\,\mu\mathrm{as}} \right)^{1/2} \times 650\ \mathrm{km}$$

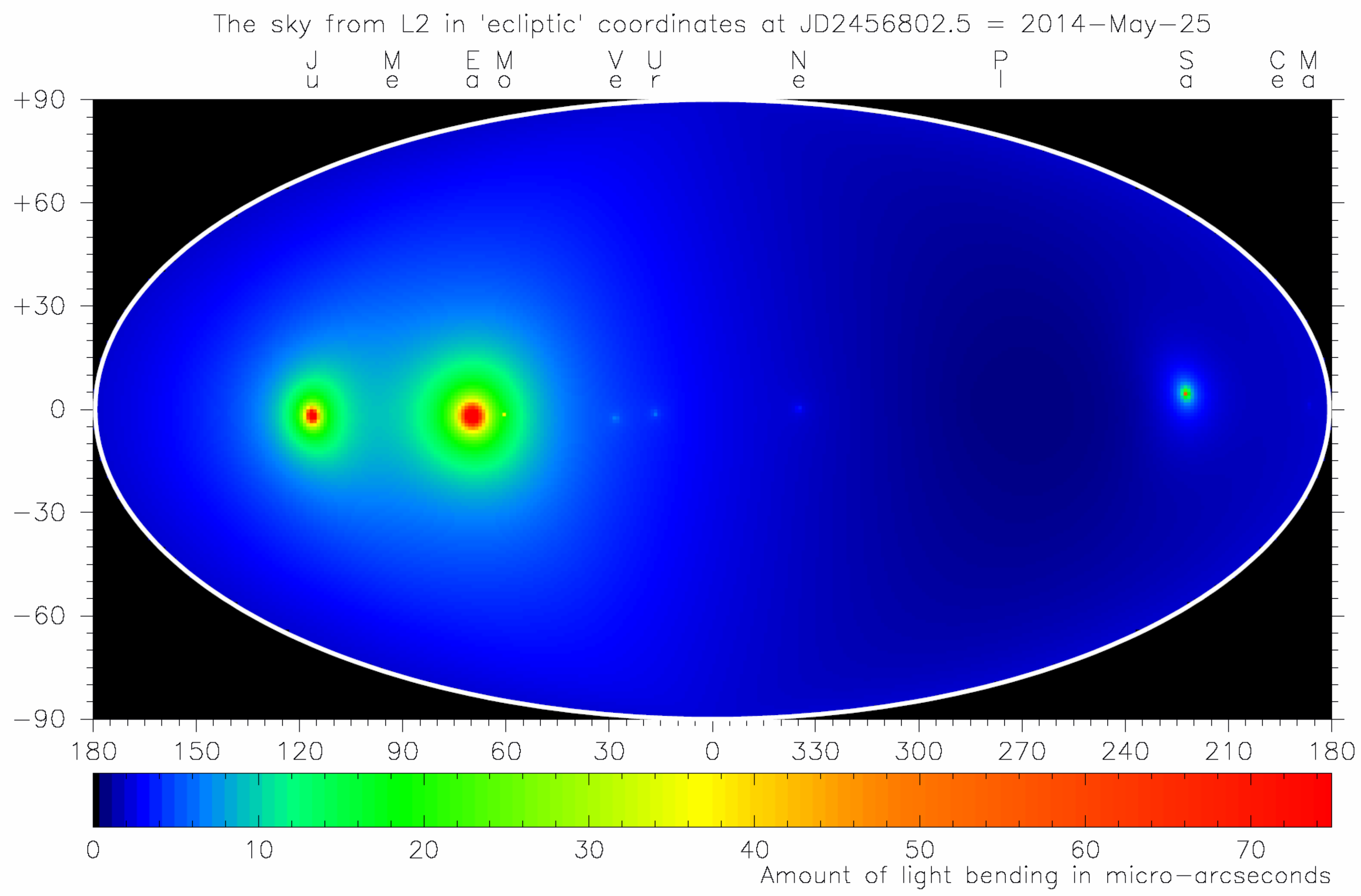



Ganymede	35
Titan	32
Io	30
Callisto	28
Triton	20
Europe	19

Pluto	7
Charon	4
Titania	3
Oberon	3
Iapetus	2
Rea	2
Dione	1
Ariel	1
Umbriel	1
Ceres	1


Gravitational light deflection:



The Barycentric Celestial Reference System

- The BCRS is suitable to model processes in the whole solar system

$$g_{00} = -1 + \frac{2}{c^2} w(t,\boldsymbol{x}) - \frac{2}{c^4} w^2(t,\boldsymbol{x}),$$

$$g_{0i} = -\frac{4}{c^3} w^i(t,\boldsymbol{x}),$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2} w(t,\boldsymbol{x})\right).$$

t = TCB

$$w(t,\boldsymbol{x}) = G\int d^3x' \frac{\sigma(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} + \frac{1}{2c^2} G \frac{\partial^2}{\partial t^2}\int d^3x' \sigma(t,\boldsymbol{x}')\,|\boldsymbol{x}-\boldsymbol{x}'|, \quad w^i(t,\boldsymbol{x}) = G\int d^3x' \frac{\sigma^i(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|},$$

$\sigma = \left(T^{00} + T^{kk}\right)/c^2, \quad \sigma^i = T^{0i}/c, \quad T^{\mu\nu}$ is the BCRS energy-momentum tensor

$$\lim_{\substack{|\mathbf{x}|\to\infty \\ t=\text{const}}} g_{\mu\nu} = \eta_{\mu\nu}$$

$\Longrightarrow$ Isolated system!

Barycentric: orientation of spatial axes

IAU-GA 2006, Prag:

orientation of spatial BCRS axes given by the ICRF

Geocentric Celestial Reference System

The GCRS is adopted by the International Astronomical Union (2000) to model physical processes in the vicinity of the Earth:

A: The gravitational field of external bodies is represented only in the form of a relativistic tidal potential.

B: The internal gravitational field of the Earth coincides with the gravitational field of a corresponding isolated Earth.

$$\boxed{\begin{aligned} G_{00} &= -1 + \frac{2}{c^2} W(T, \boldsymbol{X}) - \frac{2}{c^4} W^2(T, \boldsymbol{X}), \\ G_{0a} &= -\frac{4}{c^3} W^a(T, \boldsymbol{X}), \\ G_{ab} &= \delta_{ab}\left(1 + \frac{2}{c^2} W(T, \boldsymbol{X})\right). \end{aligned}}$$

- In the local A-frame: the local metric $W^\alpha \equiv (W, W^a)$ is split into self- and external-part

$$W^\alpha = W^{+\alpha} + \overline{W}^\alpha$$

- self-part is expanded in terms of "physical" mass- and current-moments: M_L, S_L

- external part determined by transformation of potentials self-part coming from the Earth itself

- In the local A-frame: the local metric $W^\alpha \equiv (W, W^a)$ is split into self- and external-part

$$W^\alpha = W^{+\alpha} + \overline{W}^\alpha$$

- self-part is expanded in terms of "physical" mass- and current-moments: M_L, S_L

- external part determined by transformation of potentials

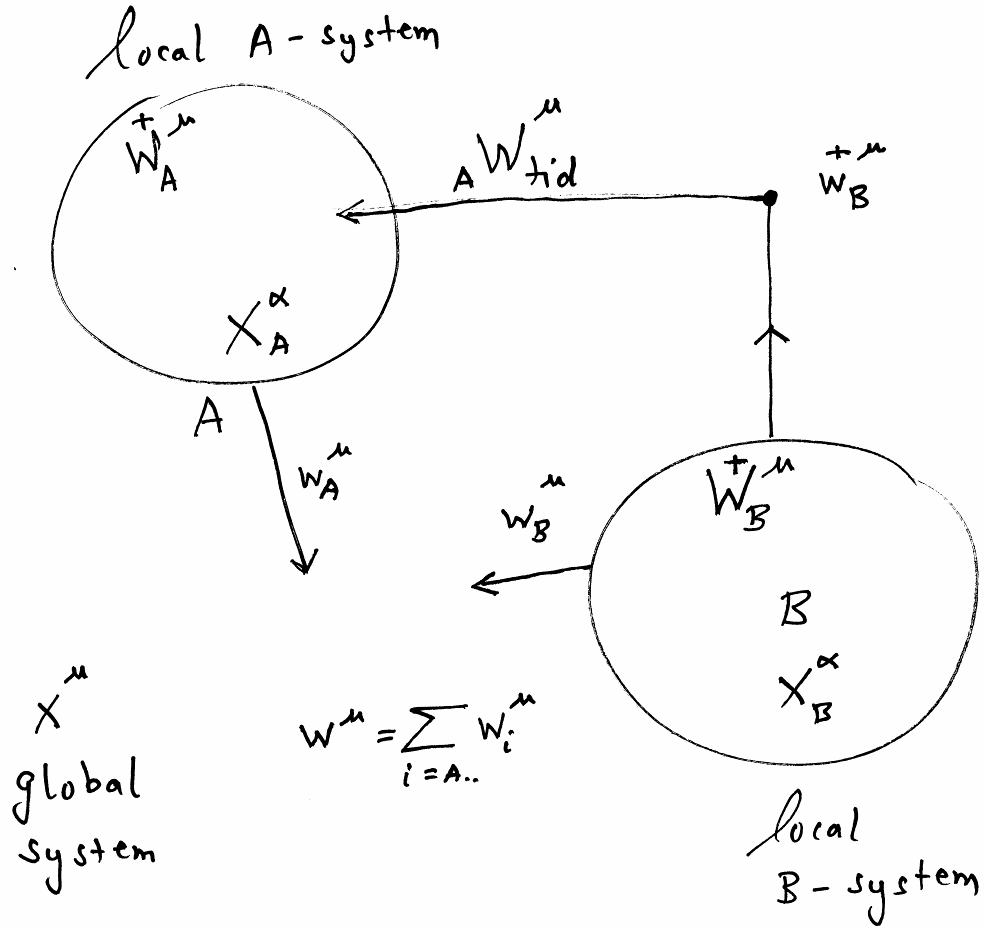
- In the local A-frame: the local metric $W^\alpha \equiv (W, W^a)$ is split into self- and external-part

$$W^\alpha = W^{+\alpha} + \overline{W}^\alpha$$

- self-part is expanded in terms of "physical" mass- and current-moments: M_L, S_L

- external part determined by transformation of potentials

External part coming from inertial effects (linear term)
and other bodies (quadratic and higher order terms)



local A-system
$\overset{+}{W}{}^{\mu}_{A}$
X^{α}_{A}
A
${}_{A}W^{\mu}_{tid}$
$\overset{+}{w}{}^{\mu}_{B}$
w^{μ}_{A}
w^{μ}_{B}
$\overset{+}{W}{}^{\mu}_{B}$
B
X^{α}_{B}
x^{μ}
global system
$w^{\mu} = \sum_{i=A..} w^{\mu}_{i}$
local B-system

- The BCRS and GCRS potentials of the central body are simply related:

$$W_\mathrm{E}(T,\boldsymbol{X}) = w_\mathrm{E}(t,\boldsymbol{x})\left(1+\frac{2}{c^2}v_\mathrm{E}^2\right) - \frac{4}{c^2}v_\mathrm{E}^i w_\mathrm{E}^i(t,\boldsymbol{x}) + O(c^{-4})$$

$$W_\mathrm{E}^a(T,\boldsymbol{X}) = R_i^a\left[w_\mathrm{E}^i(t,\boldsymbol{x}) - v_\mathrm{E}^i w_\mathrm{E}(t,\boldsymbol{x})\right] + O(c^{-2})$$

- Having the structure of the GCRS potentials one can easily restore the the structure of the BCRS potentials…

Theorem: In any local system A the potentials $W^{+}{}^{A}_{\alpha}(X^{\beta})$ admit, everywhere outside body A, the following multipole expansion (harmonic gauge)

$$\overset{+}{W}{}^{A}(T,\boldsymbol{X}) = G\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left[R^{-1}\boldsymbol{M}^A_L\,(T\pm R/c)\right] + O(4)$$

$$\overset{+}{W}{}^{A}_{a}(T,\boldsymbol{X}) = -G\sum_{l\geq 1}\frac{(-)^l}{l!}\left\{\partial_{L-1}\left[R^{-1}\frac{d}{dT}\boldsymbol{M}^A_{aL-1}\right] + \right.$$

$$\left. + \frac{l}{l+1}\epsilon_{abc}\partial_{bL-1}\left[R^{-1}\boldsymbol{S}^A_{cL-1}\right]\right\} + O(2)$$

with

$$M^A_L(T\pm R/c) \equiv \frac{1}{2}\left[M^A_L(T+R/c) + M^A_L(T-R/c)\right]$$

In the expansion of the exterior gravitational fields we face two families of multipole moments:

M_L : mass-moments
S_L: spin-moments

M_L are equivalent to potentials coefficients (C_{lm}, S_{lm})

M.Panhans: works on models for bodies with higher spin-moments
J.Meichsner: works on physical effects outside bodies with higher spin-moments

Instead of expansion in terms of spherical harmonics one works with expansions in terms of Cartesian symmetric and trace-free (STF) tensors

Let T_L **be some Cartesian tensor;** $\longrightarrow$ $L = i_1 \cdots i_l$

$i = 1, 2, 3$

$= x, y, z$

$$T_{(L)} \equiv T_{(i_1 \ldots i_l)} = \frac{1}{l!} \sum_\pi T_{i_{\pi(1)} \ldots i_{\pi(l)}}$$

$$\hat{T}_L \equiv STF(T_L)$$

Example: $N_i = X^i/R$; $N^L \equiv N_{i_1} \ldots N_{i_l}$

$$R^2\, \hat{N}_{ij} = X^i X^j - \frac{1}{3} R^2 \delta_{ij}$$

One finds

$$\partial_L \left(\frac{1}{R}\right) = \hat{\partial}_L \left(\frac{1}{R}\right) = (-1)^l (2l-1)!! \frac{\hat{N}_L}{R^{l+1}}$$

$(2l-1)!! = (2l-1)(2l-3) \ldots (2 \text{ or } 1)$

Relativistic Celestial Mechanics

PHYSICAL REVIEW D VOLUME 43, NUMBER 10 15 MAY 1991

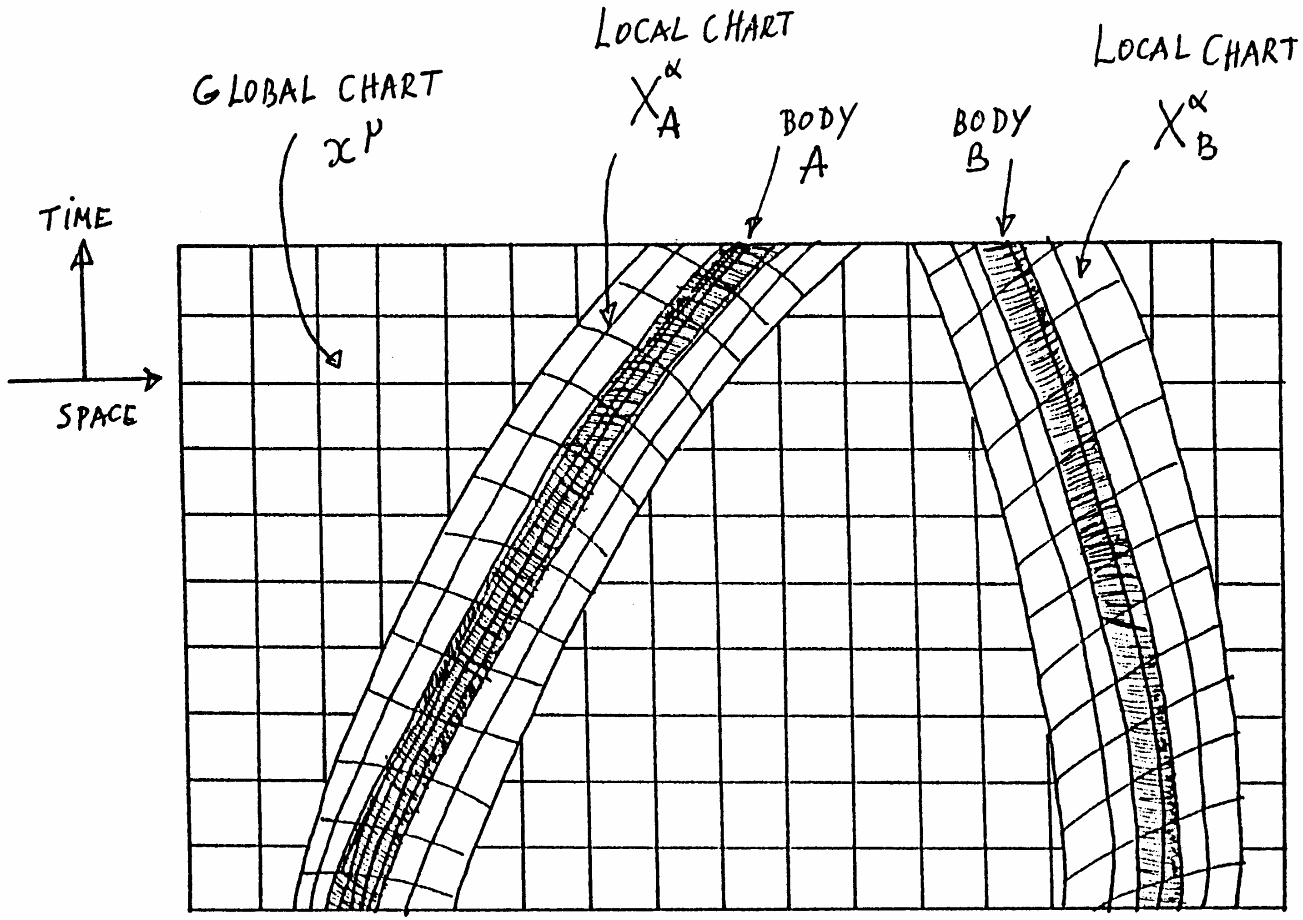
General-relativistic celestial mechanics. I. Method and definition of reference systems

Thibault Damour
Institut des Hautes Etudes Scientifiques, 91440 Bures sur Yvette, France
and Département d'Astrophysique Relativiste et de Cosmologie, Observatoire de Paris,
Centre National de la Recherche Scientifique, 92195 Meudon CEDEX, France

Michael Soffel and Chongming Xu*
Theoretische Astrophysik, Universität Tübingen, Auf der Morgenstelle 10, 7400 Tübingen,
Federal Republic of Germany
(Received 2 November 1990)

We present a new formalism for treating the general-relativistic celestial mechanics of systems of N arbitrarily composed and shaped, weakly self-gravitating, rotating, deformable bodies. This formalism is aimed at yielding a complete description, at the first post-Newtonian approximation level, of (i) the global dynamics of such N-body systems ("external problem"), (ii) the local gravitational structure of each body ("internal problem"), and, (iii) the way the external and the internal problems fit together ("theory of reference systems"). This formalism uses in a complementary manner $N+1$ coordinate charts (or "reference systems"): one "global" chart for describing the overall dynamics of the N bodies, and N "local" charts adapted to the separate description of the structure and environment of each body. The main tool which allows us to develop, in an elegant manner, a constructive theory of these $N+1$ reference systems is a systematic use of a particular "exponential" parametrization of the metric tensor which has the effect of linearizing both the field equations, and the transformation laws under a change of reference system. This linearity allows a treatment of the first post-Newtonian relativistic celestial mechanics which is, from a structural point of view, nearly as simple and transparent as its Newtonian analogue. Our scheme differs from previous attempts in several other respects: the structure of the stress-energy tensor is left completely open; the spatial coordinate grid (in each system) is fixed by algebraic conditions while a convenient "gauge" flexibility is left open in the time coordinate [at the order $\delta t = O(c^{-4})$]; the gravitational field locally generated by each body is skeletonized by particular relativistic multipole moments recently introduced by Blanchet and Damour, while the external gravitational field experienced by each body is expanded in terms of a particular new set of relativistic tidal moments. In this first paper we lay the foundations of our formalism, with special emphasis on the definition and properties of the N local reference systems, and on the general structure and transformation properties of the gravitational field. As an illustration of our approach we treat in detail the simple case where each body can, in some approximation, be considered as generating a spherically symmetric gravitational field. This "monopole truncation" leads us to a new (and, in our opinion, improved) derivation of the Lorentz-Droste-Einstein-Infeld-Hoffmann equations of motion. The detailed treatment of the relativistic motion of bodies endowed with arbitrary multipole structure will be the subject of subsequent publications.

For the gravitational N body system one introduces N+1 RS



One global and N local coordinate systems are used for the description of the gravitational N-body system

Equations derived from

- relating the local coordinates $(z^a(T))$ with corresponding body

$$M_a = \frac{dM_a}{dT} = \frac{d^2 M_a}{dT^2} = 0$$

[z^a: PN center of mass for all times]

Local form

$$0 = \frac{d^2 M_a}{dT^2} = \sum_{l \geq 0} \frac{1}{l!} M_L G_{La} + (c^{-2} - \text{terms})$$

Since

$$G_a = -\frac{d^2 z^a}{dt^2} + \overline{W}_{,a}|_{X^a=0} + (c^{-2} - \text{terms})$$

Global form

$$\frac{d^2 z^a}{dt^2} = \overline{W}_{,a}| + (c^{-2} - \text{terms})$$

obtained fully to PN-order in terms of $M_L, S_L; G_L, H_L$

Monopole limit without spins $\longrightarrow$ EIH-equations of motion

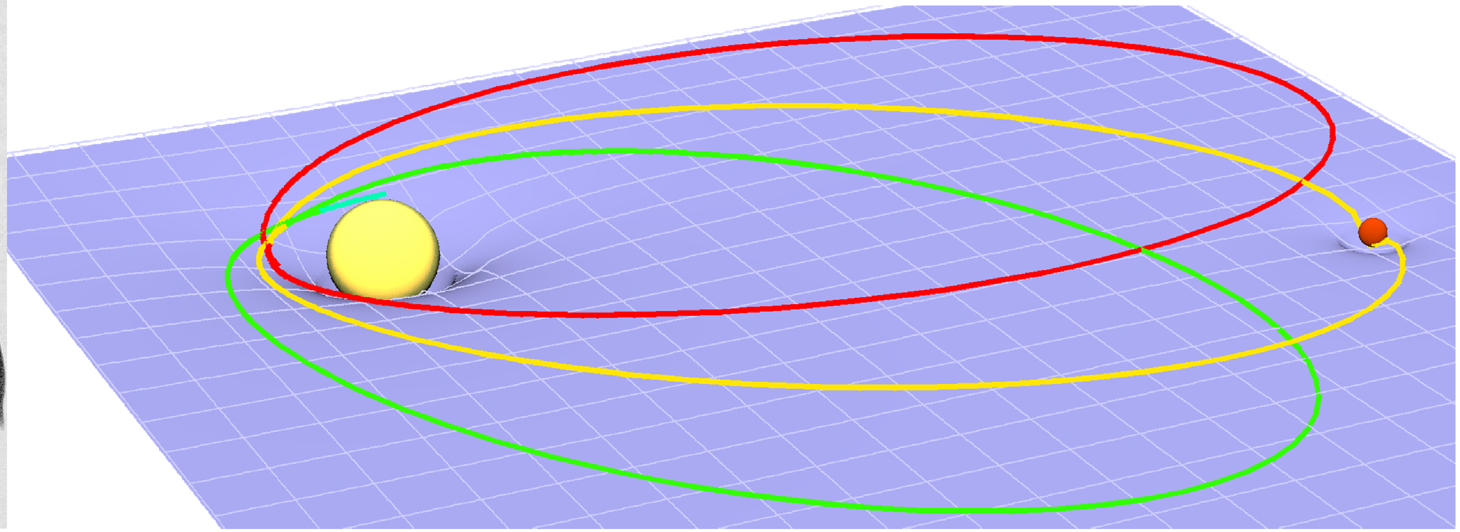
Equations of translational motion

- The equations of translational motion (e.g. of a planet) in the BCRS

$$g_{00} = -1 + \frac{2}{c^2} w(t,\boldsymbol{x}) - \frac{2}{c^4} w^2(t,\boldsymbol{x}),$$
$$g_{0i} = -\frac{4}{c^3} w^i(t,\boldsymbol{x}),$$
$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2} w(t,\boldsymbol{x})\right).$$

- The equations coincide with the well-known Einstein-Infeld-Hoffmann (EIH) equations in the corresponding limit

$$\ddot{\boldsymbol{x}}_A = -\sum_{B\neq A} GM_B \frac{\boldsymbol{x}_A - \boldsymbol{x}_B}{|\boldsymbol{x}_A - \boldsymbol{x}_B|^3} + \frac{1}{c^2}\boldsymbol{F}(t)$$



EIH equations rederived

- Assumption:
  each body is a mass monopole in its own local reference system

$$\mathscr{M}_L = 0 \text{ for } l \geq 1\,, \quad \mathscr{S}_L = 0 \text{ for } l \geq 1$$

1. Transform the GCRS potentials into the BCRS potentials
2. derive EOM from M_a = 0

Details: DSX I. Output: the usual EIH EOM:

$$\ddot{x}_A = -\sum_{B\neq A} \frac{GM_B}{r_{AB}^2}\mathbf{n}_{AB}\left[1+\frac{1}{c^2}\left\{\mathbf{v}_A^2+2\mathbf{v}_B^2-4\mathbf{v}_A\cdot\mathbf{v}_B-\frac{3}{2}(\mathbf{n}_{AB}\cdot\mathbf{v}_B)^2\right.\right.$$

$$\left.\left.-4\sum_{C\neq A}\frac{GM_C}{c^2 r_{AC}}-\sum_{C\neq B}\frac{GM_C}{c^2 r_{BC}}\left(1+\frac{1}{2}\frac{r_{AB}}{r_{CB}}\mathbf{n}_{AB}\cdot\mathbf{n}_{CB}\right)\right\}\right]$$

$$-\frac{7}{2}\sum_{B\neq A}\sum_{C\neq B}\mathbf{n}_{BC}\frac{G^2 M_B M_C}{c^2 r_{AB} r_{BC}^2}+\sum_{B\neq A}(\mathbf{v}_A-\mathbf{v}_B)\frac{GM_B}{c^2 r_{AB}^2}(4\mathbf{n}_{AB}\cdot\mathbf{v}_A-3\mathbf{n}_{AB}\cdot\mathbf{v}_B)$$

Dynamically and kinematically non-rotating reference systems

Kinematically and dynamically non-rotating

- GCRS Potentials $W(T,\boldsymbol{X}) = W_E(T,\boldsymbol{X}) + Q_a(T)X^a + W_T(T,\boldsymbol{X}),$

$$W^a(T,\boldsymbol{X}) = W_E{}^a(T,\boldsymbol{X}) + \underline{\frac{1}{2}\varepsilon_{abc}C_b(T)X^c} + W_T{}^a(T,\boldsymbol{X})$$

- Coordinate transformations BCRS-GCRS:

$$T = t - \frac{1}{c^2}\left(A(t) + v_E^i r_E^i\right) + \frac{1}{c^4}\left(B(t) + B^i(t)r_E^i + B^{ij}(t)r_E^i r_E^j + C(t,\boldsymbol{x})\right) + O\left(c^{-5}\right),$$

$$X^a = \underline{R_i^a(t)}\left(r_E^i + \frac{1}{c^2}\left(\frac{1}{2}v_E^i v_E^j r_E^j + D^{ij}(t)r_E^j + D^{ijk}(t)r_E^j r_E^k\right)\right) + O\left(c^{-4}\right)$$

$$r_E^i = x^i - x_E^i(t)$$

$x_E^i(t)$ and $v_E^i(t)$ are the BCRS position and velocity of the Earth

$R_i^a(t)$ is an orthogonal (rotation) matrix,

Kinematically and dynamically non-rotating

$C_a(T)$ defines rotational motion of the spatial axes of GCRS

$C_a(T) = 0 \quad \Longrightarrow$

no Coriolis forces in the equations of motion
of a test particle in the GCRS;
dynamically non-rotating GCRS

$\dot{R}_i^a(t) = 0 \quad \Longrightarrow$ No spatial rotation between GCRS and BCRS;
kinematically non-rotating GCRS

- The standard choice for astronomical data processing is the kinematically non-rotating GCRS: $R^a_i \equiv \delta_{ia}$

Thus: the orientation of spatial GCRS axes is determined by the orientation of BCRS axes (i.e., by the ICRF)

- Coriolis forces in the GCRS equations of motion, e.g. for satellites

$$C_a = -\frac{1}{2} c^2 R^i_a \left(\Omega^i_{GP} + \Omega^i_{LTP} + \Omega^i_{TP} \right),$$

geodetic $\quad \Omega^i_{GP} = -\frac{3}{2} c^{-2} \varepsilon_{ijk} v^j_E \overline{w}_{,k} (\boldsymbol{x}_E) \quad$ 1.92″/cy + 0.150 mas

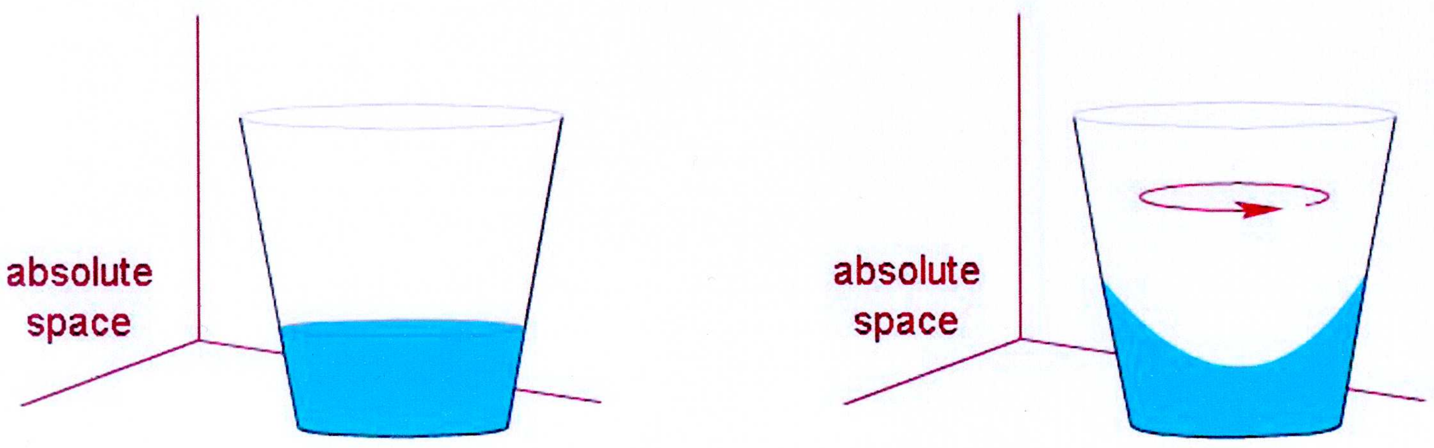
Lense-Thirring $\quad \Omega^i_{LTP} = -2c^{-2} \varepsilon_{ijk} \overline{w}^{j,k} (\boldsymbol{x}_E) \quad$ 2 mas/cy

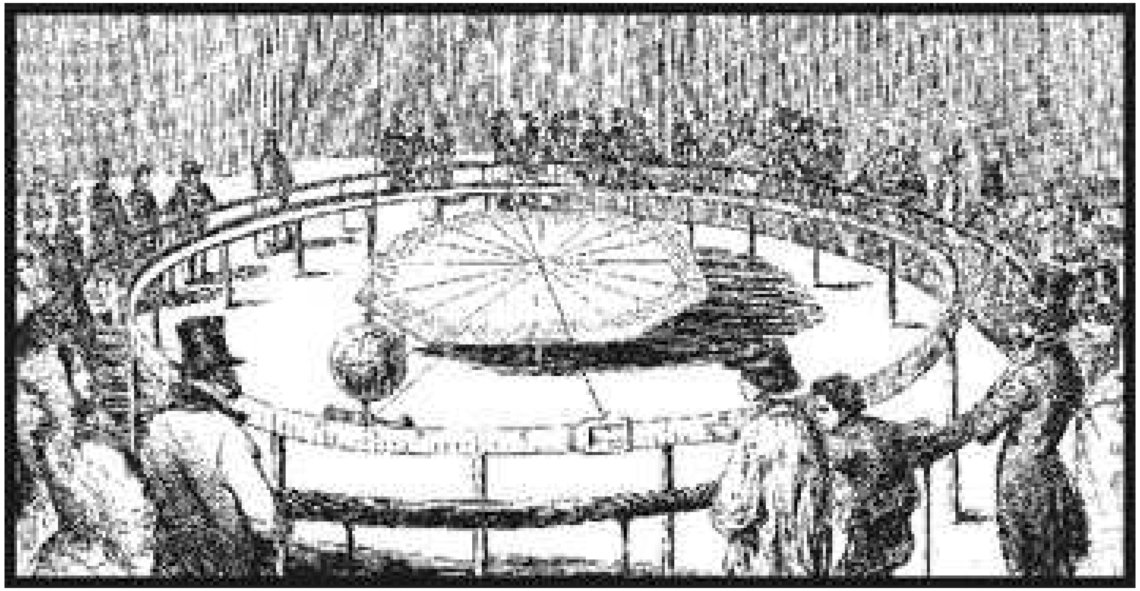
Thomas $\quad \Omega^i_{TP} = -\frac{1}{2} c^{-2} \varepsilon_{ijk} v^j_E R^k_a Q_a \quad$ 0.004 μas/cy

The problem of inertia in GRT

Inertial frames

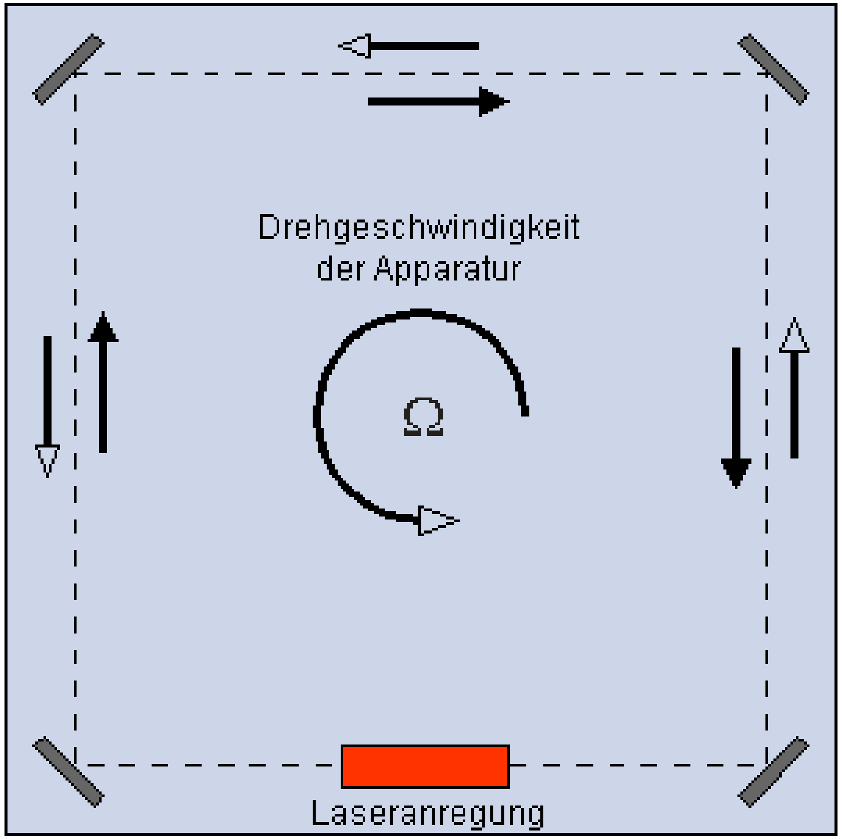
in Newton‘s theory inertial frames are determined by absolute space





Leon Foucault, 1851; Pantheon, Paris

A modern version of the Foucault pendulum



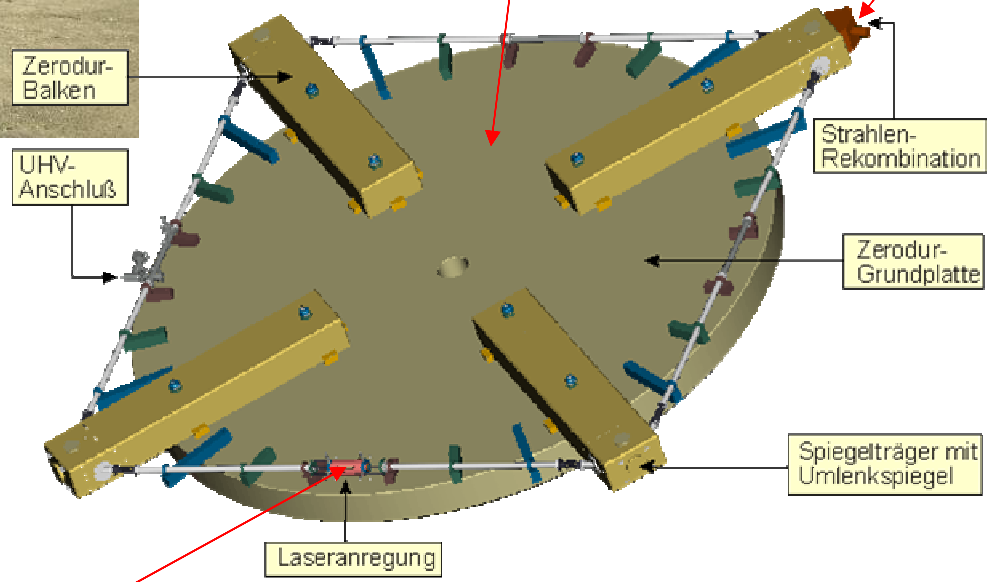
a laser gyro

$$\Delta \nu_{Sagnac} = \frac{4\mathbf{A} \cdot \Omega}{\lambda P}$$

The laser-gyro in Wettzell, Germany

Cerodur groundplate 16 m^2

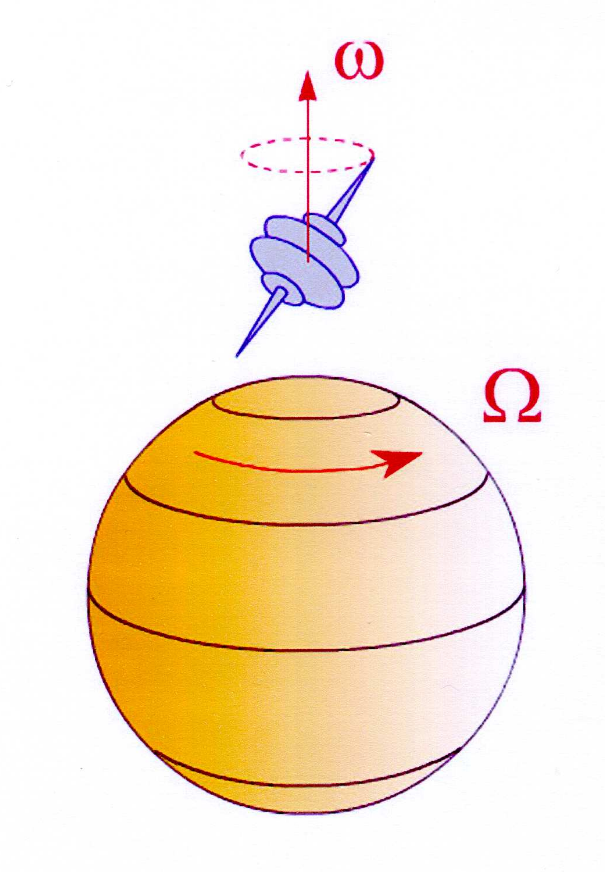
beam recombiner



laser excitation

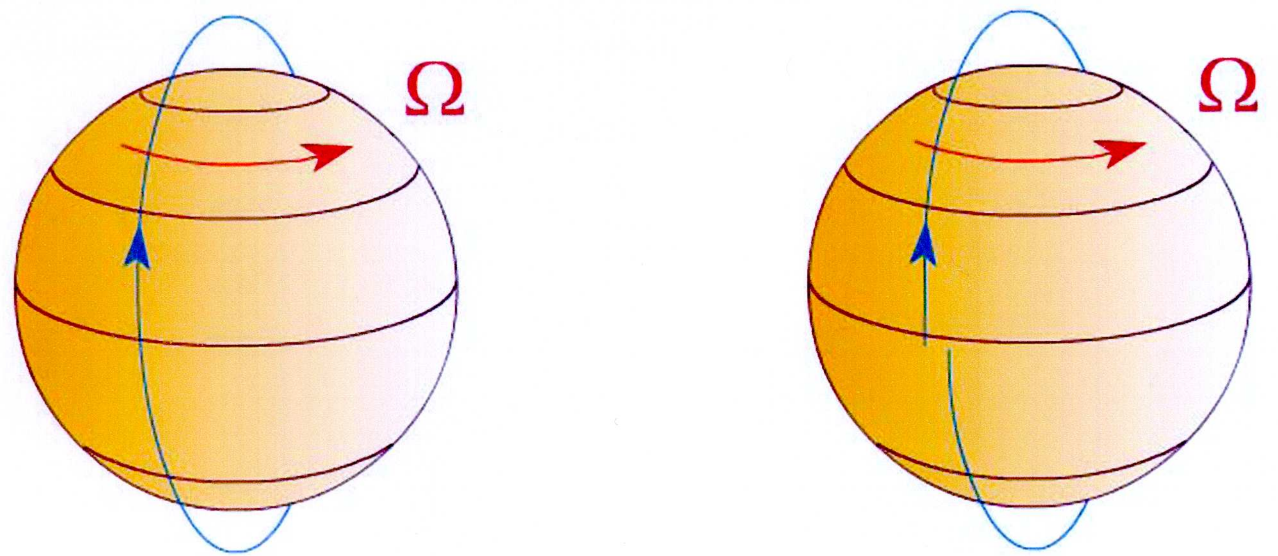
Dragging of inertial frames

In GRT locally inertial systems rotate with respect to the fixed stars



A torque free gyro is dragged by the rotating Earth

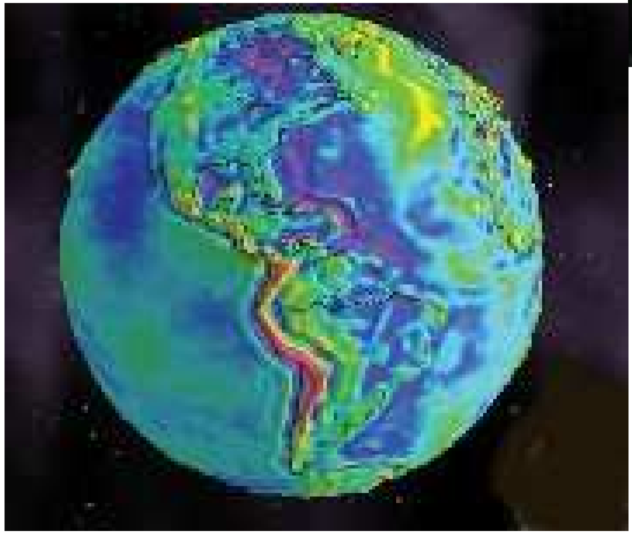
(Lense-Thirring effect)



Lense-Thirring effect in the motion of satellites:
precession of orbit in space

Frame dragging

Experimentally detected in the motion of satellites
by I.Ciufolini



Lageos I (II)

nodal drift:
20 μas/rev.

Ignazio Ciufolini

The geodetic precession

A torque-free gyro, moving with the Earth precesses w.r.t. the quasar-sky because of its motion about the Sun.

This **geodetic precession** amounts to

$$\Omega_{GP} = (3/2c^2)\,\mathbf{v}_E \times \nabla U_{ext} \approx 1.98\ \text{''/cen.}$$

If the Earth is considered in rotation w.r.t. the GCRS
the geodetic precession/nutation
will be in the PN-matrix (even for zero ellipticity!)

For more details on ST reference systems see:

Contents

PLUS:

a large number of

- solved Exercises

- Maple programs

FIN