ABOUT THE MACCULLAGH RELATIONS IN RELATIVITY

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ABSTRACT. The Newtonian MacCullagh relations relate two different aspects of the global Earth: its rotational motion and its external gravitational field. They relate components of the tensor of inertia with $l = 2$ potential coefficients.These two concepts can be generalized to the first post-Newtonian approximation to Einstein's theory of gravity and one can ask if the usual form of the MacCullagh relation still holds if $1/c^2$ are taken into account. To answer this question a simple Newtonian model for the Earth was employed: a uniformly rotating homogenous oblate spheroid. For this model a violation of the usual MacCullagh relations was found. Implications for a post-Newtonian nutation model are shown to be negligible.

1. THE NEWTONIAN MACCULLAGH RELATIONS

The Newtonian MacCullagh relations relate two different aspects of the global Earth: its rotational motion and its external gravitational field. More precisely they relate the components of the tensor of inertia tensor with $l = 2$ potential coefficients. If the external gravity field of the Earth is expanded in terms of spherical harmonics in the form

$$U(\mathbf{x}) = \left(\frac{GM}{r}\right) \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{a}{r}\right)^l P_{lm}(\cos\theta)\big(C_{lm}\cos m\phi + S_{lm}\sin m\phi\big)$$

the Newtonian MacCullagh relations read

$$\begin{aligned} C_{21} &= -\frac{I_{13}}{Ma^2}\,, \qquad S_{21} = -\frac{I_{23}}{Ma^2}\,, \\ C_{22} &= \frac{I_{22} - I_{11}}{4Ma^2}\,, \qquad S_{22} = -\frac{I_{12}}{2Ma^2}\,, \\ C_{20} &= \frac{1}{Ma^2}\left(\frac{I_{11} + I_{22}}{2} - I_{33}\right)\,, \end{aligned}$$

where I_{ij} are the components of the moment of inertia tensor

$$I_{ij} = \int_B (\mathbf{x}^2 \delta_{ij} - x^i x^j)\rho\, d^3x\,.$$

If the external gravitational field is expanded in terms of Cartesian symmetric and trace-free (STF) tensors all five MacCullagh relations are contained in the relation ($I_{kk} = I_{11} + I_{22} + I_{33}$)

$$M_{ij} = -\mathrm{STF}(I_{ij}) = \frac{1}{3}\delta_{ij} I_{kk} - I_{ij}\,, \tag{1}$$

i.e., the Cartesian mass quadrupole tensor M_{ij} equals minus the (symmetric) trace-free part of the moment of inertia tensor. In the STF language the external gravity field is expanded as

$$U(\mathbf{x}) = G \sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} M_L \frac{\hat{n}_L}{r^{l+1}}$$

with

$$\hat{n}_L \equiv \frac{\hat{x}^L}{r^l}\,, \qquad \hat{x}^L \equiv \hat{x}^{i_1 \dots i_l} = \mathrm{STF}_{i_1 \dots i_l}\left(x^{i_1} \cdots x^{i_l}\right)$$

and every index i runs over 1,2,3 corresponding to x, y, z. E.g., $\hat{x}^{ij} = \mathrm{STF}_{i,j}(x^i x^j) = x^i x^j - \delta_{ij}\,\mathbf{x}^2/3$.

2. RELATIVITY AND THE MACCULLAGH RELATIONS

With respect to some relativistic precession/nutation theory (e.g., Klioner et al., 2001, 2010) we now want to ask about relations that might generalize the Newtonian MacCullagh relations in Relativity. Unfortunately in Einstein's theory of gravity neither a mass quadrupole tensor nor some moment of inertia tensor can be defined for a real astronomical body. However, if one resorts to the first post-Newtonian approximation to Einstein's theory of gravity both concepts can be defined. As for the mass quadrupole tensor the Blanchet-Damour quadrupole mass moment (Damour et al., 1991)

$$M_{ij} = \int_B d^3x \hat{x}_{ij} \sigma + \frac{1}{14c^2} \frac{d^2}{dt^2} \int_B d^3x \hat{x}_{ij} \mathbf{x}^2 \sigma - \frac{20}{21c^2} \frac{d}{dt} \int_B d^3x \hat{x}^{ijk} \sigma^k$$

generalizes the $l = 2$ potential coefficients. Here, $\sigma = (T^{00} + T^{ss})/c^2$, $\sigma^i = T^{0i}/c$ and $T^{\alpha\beta}$ are the components of the energy-momentum tensor (generalizing the density ρ). In Damour et al., (1993) a post-Newtonian spin vector was defined for the Earth from which a post-Newtonian moment of inertia tensor can be derived (see also Klioner 1995).

One can then ask if relation (1) is still valid for the corresponding post-Newtonian quantities. In the general case of a gravitational N-body problem this question is not so easy to answer since gravitational fields from external bodies (Moon, Sun, etc.) enter the mass quadrupole and moment of inertia tensors of the Earth. We have stated to answer our basic question by considering a simplified situation: we employed a specific Newtonian model for the $1/c^2$ terms, a uniformly rotating homogeneous oblate spheroid with semi-major axis a, semi-minor axis b (pointing in z-direction) and eccentricity e given by $e^2 = 1 - b^2/a^2$. The internal gravity field that is needed to compute various integrals can be found in Chandrasekhar (1987). For our model a violation of the usual MacCullagh relations was found. Instead we have

$$M_{ij} = -\mathrm{STF}(I_{ij}) - (Ma^2)\delta \mathrm{diag}(1, 1, -2) + \mathcal{O}(e^4)\,, \qquad (2)$$

where

$$\delta = \frac{418}{875} \; \frac{GM}{c^2 a} \, e^2\,.$$

For the Earth $\delta = 0.787 \times 10^{-12}$. To assess the consequences of the δ-term for a relativistic nutation theory we employed the numerical code that solves for the three Euler angles ϕ, ψ and ω (the obliquity) for the post-Newtonian model of rigidly rotating multipoles (Klioner et al., 2010) including the relativistic torques from Moon, Sun and planets. We found that effects from the δ-term do not exceed $10^{-6}\,\mu$as for all three Euler angles (see Fig. 1). At least for our simple model the detected violation of the usual MacCullagh relations can be neglected in the post-Newtonian precession/nutation model in the near future. For the real Earth additional aspects come into play: deviations from axial symmetry or gravitational fields from external bodies. Such effects are presently under investigations. Details will be published elsewhere.

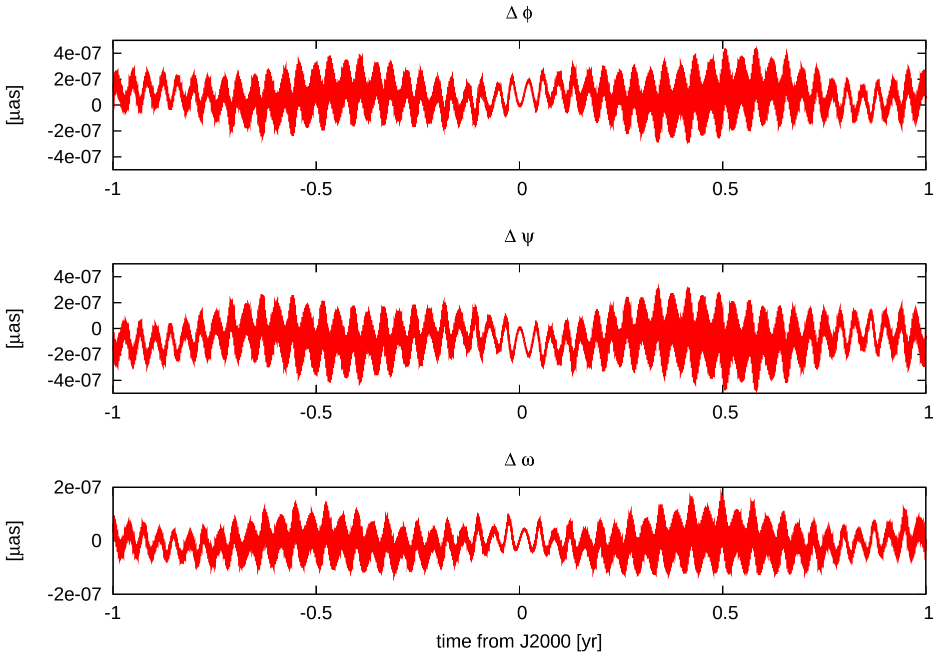

Figure 1: The effect of the relativistic violation of the MacCullagh relations over 2 years as differences in the Euler angles describing the orientation of the Earth. The main periods are 1 year and 0.5 months. The maximal amplitudes of the effect over a few hundred years around J2000 does not exceed $5 \cdot 10^{-7}\mu$as for phi and psi and $2 \cdot 10^{-7}\mu$as for omega.

3. REFERENCES

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