ON SOLUTION OF THE THREE–AXIAL EARTH'S ROTATION PROBLEM

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ABSTRACT. The three–axial rigid–body Earth's rotation problem is treated in the form compatible with the General Planetary Theory GPT. This paper completes the results of (Brumberg and Ivanova, 2007, 2010).

1. THE SYSTEM OF THE EQUATIONS FOR THE PLANETARY AND LUNAR MOTIONS AND THE EARTH'S ROTATION

The complete system of the equations for the planetary and lunar motions and the Earth's rotation was derived in (Brumberg and Ivanova, 2010). It has the form

$$\dot{X} = \mathrm{i}\mathcal{N}[PX + R(X,t)] \tag{1}$$

where

$$X = (a,\, \bar{a},\, b,\, \bar{b},\, X_{37}, \dots, X_{43}), \quad R = (R_1, \dots, R_4,\, R_{37}, \dots, R_{43}) \tag{2}$$

are vectors with 43 components (eccentric and oblique Laplace–type variables of the planets and the Moon a, b and R_i for $i = 1, 2, 3, 4$ are 9–vectors, $X_{36+\kappa}$ and $R_{36+\kappa}$ for $\kappa = 1, 2, \dots, 7$ are connected with the Earth's rotation and depend on four Euler parameters (replacing three classical Euler angles) and three components of the Earth's rotation angular velocity.

$\mathcal{N}$ and P are 43×43 diagonal matrices of the structure

$$\mathcal{N} = \mathrm{diag}(N, N, N, N, n, n, n, n, n, n, n)\,,$$

$$P = \mathrm{diag}(E_{(9)}, -E_{(9)}, E_{(9)}, -E_{(9)}, 1,\, -1,\, 1,\, -1, -4\sqrt{k_1 k_2}\,, 4\sqrt{k_1 k_2}\,, 0)$$

where N is 9×9 diagonal matrix of mean motions n_i, $E_{(9)}$ is unitary matrices of dimension 9×9, k_1, k_2 and n are determined by

$$k_1 = \frac{I_3 - I_1}{2I_2}\,, \qquad k_2 = \frac{I_3 - I_2}{2I_1}\,, \qquad n = -\frac{\Omega}{2}\,,$$

$\Omega = 7.292115 \cdot 10^{-5}$ rad/s being the mean Earth's rotation velocity and $I_i (i = 1, 2, 3)$ being principal inertia moments.

2. THE SECULAR SYSTEM

Our aim is to reduce (1) to the secular system. For that, the system (1) is subjected to a number of the normalizing Birkhoff and the linear transformations. As a result, the secular system describing the evolution of the Earth's rotation (depending on the planetary and lunar evolution) is presented by

$$\dot{p}_1 = \mathrm{i}\, n(p_1 + F_{37}), \tag{3}$$

$$\dot{p}_3 = \mathrm{i}\, n(p_3 + F_{39}), \tag{4}$$

$$\dot{p}_5 = \mathrm{i}\, n(-4\sqrt{k_1 k_2}\, p_5 + F_{41}), \tag{5}$$

$$\dot{p}_7 = \mathrm{i}\, n\, F_{43} \tag{6}$$

with conjugate equations for p_1, p_3, p_5. Our computations show that the right–hand members F consist of two parts, i.e.

$$F_{36+\kappa} = U_\kappa^{*(1)} + U_\kappa^{*(2)}\,, \qquad \kappa = 1, \ldots, 7 \tag{7}$$

with

$$U_\kappa^{*(1)} = p_\kappa(1-\delta_{\kappa 7}) \sum\nolimits^{*} U_{iklm}^{(\kappa,s)} (p_1p_2)^{s_2}(p_3p_4)^{s_4}(p_5p_6)^{s_6}p_7^{s_7} \times (w_N\overline{w}_N)^{s_4} \prod_{j=1}^{9}(z_j\overline{z}_j)^{k_j}(w_j\overline{w}_j)^{m_j}\,, \tag{8}$$

$$U_\kappa^{*(2)} = (1-\delta_{\kappa 7})p_5^{\delta_{\kappa 5}}p_6^{\delta_{\kappa 6}} \sum\nolimits^{*} U_{iklm}^{(\kappa,s)} \left(\prod_{j=1}^{4} p_j^{s_j}\right)(p_5p_6)^{\min\{s_5,s_6\}}p_7^{s_7}\times$$

$$\times(w_N\bar{w}_N)^{\max\{s_3-\delta_{\kappa 3},\, s_4-\delta_{\kappa 4}\}} \prod_{j=1}^{9}(z_j\overline{z}_j)^{k_j}(w_j\overline{w}_j)^{m_j} \tag{9}$$

with numerical coefficients $U_{iklm}^{(\kappa,s)}$. δ_{ij} is the Kronecker symbol. $p_2 = \overline{p}_1, p_4 = \overline{p}_3, p_6 = \overline{p}_5$. It is seen from (8)–(9) that F_{43} is equal to 0 and, therefore, the equation (6) can be omitted ($p_7 = const$).

Equations (3)–(6) with the right–hand members (7)–(9) admit three first integrals

$$p_1p_2 + (w_N\bar{w}_N)p_3p_4 = C_1, \qquad p_5p_6 = C_2, \qquad p_7 = C_3 \tag{10}$$

with real constants $C_1\,, C_2\,, C_3$.

The secular system describing the evolution of the planetary and lunar orbits (independent of the Earth's rotation) may be presented in the form

$$\dot{z}_\sigma = \mathrm{i}(\mu_\sigma z_\sigma + n_\sigma U_{1\sigma}^*), \qquad \dot{w}_\sigma = \mathrm{i}(\nu_\sigma w_\sigma + n_\sigma U_{3\sigma}^*) \tag{11}$$

with

$$U_{\kappa\sigma}^* = (z_\sigma\delta_{\kappa 1} + w_\sigma\delta_{\kappa 3})\sum\nolimits^{*} U_{iklm}^{(\kappa,\sigma)}(z_j\overline{z}_j)^{k_j}(w_j\overline{w}_j)^{m_j}\,, \quad \kappa = 1, 3\,, \quad \sigma = 1, 2, \ldots, 9\,,$$

μ_j and ν_j being the planetary ($j = 1, 2, \ldots, 8$) and lunar ($j = 9$) motions of pericentres and nodes, respectively and ν_N being 0 (in our computations $N = 5$). This system admits the first integrals

$$z_j\overline{z}_j = \mathrm{const}\,, \qquad w_j\overline{w}_j = \mathrm{const} \tag{12}$$

leading to straightforward integration.

3. SOLUTION OF THE EARTH'S ROTATION SECULAR SYSTEM

Designating $p_1 = g,\ p_3 = h,\ p_5 = f$ one may present the secular system for the Earth's rotation in the form

$$\begin{aligned}
\dot{g} &= \mathrm{i}\,n\big[gG(g\overline{g},\, h\overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j) + \Phi(g,\, \overline{g},\, h,\, \overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j)\big]\,,\\
\dot{h} &= \mathrm{i}\,n\big[hH(g\overline{g},\, h\overline{h},\, z_j\overline{z}_j,\, wj\overline{w}_j) + \Psi(g,\, \overline{g},\, h,\, \overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j)\big]\,,\\
\dot{f} &= \mathrm{i}\,n\big[fF(g\overline{g},\, h\overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j) + \Theta(g,\, \overline{g},\, h,\, \overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j)\big]
\end{aligned} \tag{13}$$

where

$$gG = g + U_1^{*(1)}\,, \quad hH = h + U_3^{*(1)}\,, \quad fF = -4\sqrt{k_1k_2}\,f + U_5^{*(1)}\,, \tag{14}$$

$$\Phi = U_1^{*(2)}\,, \qquad \Psi = U_3^{*(2)}\,, \qquad \Theta = U_5^{*(2)}\,. \tag{15}$$

The third equation of (13) is separated from the first two. Hence, these equations may be treated analytically in much the same manner as in (Brumberg and Ivanova, 2007). It is seen from (8)–(9) that the planetary and lunar coordinates enter in every part as the functions of the first integrals (12), i.d. they enter as the constants. Therefore, with taking into account the first integrals (10) the first parts $U_\kappa^{*(1)}$ of the right–hand members are ready to integrate (13), the second parts $U_\kappa^{*(2)}$ are of more general form but they are less significant than the first parts. Our computations show that within the linear theory with respect to small parameters depending on the dynamical flattenings and C_3

$$G = 1 + (g\overline{g} - h\overline{h}\,w_N\overline{w}_N)h\overline{h}\,w_N\overline{w}_N\,\Sigma_1 + C_0\,, \tag{16}$$

$$H = 1 - (g\overline{g} - h\overline{h}\, w_N\overline{w}_N) g\overline{g}\,\Sigma_1 + C_0\,, \tag{17}$$

$$F = -4\sqrt{k_1 k_1} - 4\sqrt{k_1 k_2} C_3 + (C_1^2 - 6g\overline{g}h\overline{h}\, w_N\overline{w}_N)\,\Sigma_1'\,, \tag{18}$$

$$\Phi = \overline{h}\, w_N\overline{w}_N \left\{ \left[g^3\overline{g} - 3g^2 h\overline{h}\, w_N\overline{w}_N - 3g\overline{g}\, h^2\, w_N\overline{w}_N + h^3\overline{h}\,(w_N\overline{w}_N)^2\right]\Sigma_2 + \right.$$
$$\left. + (\overline{g}h^3\, w_N\overline{w}_N - g^3\overline{h})\, w_N\overline{w}_N\Sigma_2' \right\}, \tag{19}$$

$$\Psi = -\overline{g}\left\{ \left[[g^3\overline{g} - 3g^2 h\overline{h}\, w_N\overline{w}_N - 3g\overline{g}h^2\, w_N\overline{w}_N + h^3\overline{h}\,(w_N\overline{w}_N)^2\right]\Sigma_2 + \right.$$
$$\left. + (\overline{g}h^3\, w_N\overline{w}_N - g^3\overline{h})\, w_N\overline{w}_N\Sigma_2' \right\}, \tag{20}$$

$$\Theta = -f\, w_N\overline{w}_N (g\overline{g} - h\overline{h}\, w_N\overline{w}_N)(g\overline{h} + \overline{g}h)\Sigma_2'' \tag{21}$$

where the constants $\Sigma_1, \Sigma_1', \Sigma_2, \Sigma_2', \Sigma_2''$ are the functions of the first integrals (12), C_0 depends on k_1, k_2 and two first integrals (10).

To solve (13) the method of the variation of the arbitrary constants is used. Neglecting temporarily the second parts one gets the trigonometrical solution

$$g = g_0 \exp \mathrm{i}\,\xi\,, \quad h = h_0 \exp \mathrm{i}\,\eta\,, \quad f = f_0 \exp \mathrm{i}\,\zeta\,,$$
$$\xi = n\Delta t + \xi_0\,, \quad \eta = n\sigma t + \eta_0\,, \quad \zeta = n\chi t + \zeta_0 \tag{22}$$

with real constants g_0, h_0, f_0, ξ_0, η_0, ζ_0 and the frequency factors

$$\Delta = G\,, \qquad \sigma = H\,, \qquad \chi = F\,. \tag{23}$$

The amplitudes g_0, h_0 of the trigonometrical solution (22) are determined from (23)

$$g_0^4 = \frac{(1 + C_0 - \sigma)^2}{[2(1 + C_0) - \Delta - \sigma]\Sigma_1}\,, \quad h_0^4 w_N^2 \overline{w}_N^2 = \frac{(\Delta - 1 - C_0)^2}{[2(1 + C_0) - \Delta - \sigma]\Sigma_1}\,.$$

The amplitude $f_0 = \sqrt{C_2}$. By combining three frequencies n, $n\Delta$, $n\sigma$ one can restore the fundamental frequencies of the classical solution.

$$n(\Delta + \sigma) = \dot{\varphi}\,, \qquad n(\Delta - \sigma) = \dot{\psi}\,. \tag{24}$$

The frequency χ corresponds to the Euler period of the Earth's rotation.

To evaluate the influence of Φ, Ψ, Θ one may retain the form (22) with constant Δ, σ, χ and slowly change g_0, h_0, f_0, ξ_0, η_0, ζ_0. By substituting (22) into (13) one gets the rates of changing these variables

$$\mathrm{i}\, g\dot{\xi}_0 + g g_0^{-1}\dot{g}_0 = \mathrm{i}\, n[\Phi + g(G - \Delta)], \tag{25}$$

$$\mathrm{i}\, h\dot{\eta}_0 + h h_0^{-1}\dot{h}_0 = \mathrm{i}\, n[\Psi + h(H - \sigma)], \tag{26}$$

$$\mathrm{i}\, f\dot{\zeta}_0 + f f_0^{-1}\dot{f}_0 = \mathrm{i}\, n[\Theta + f(F - \chi)], \tag{27}$$

By combining these equations with their conjugates one obtains at once

$$2g_0\dot{g}_0 = \mathrm{i}\, n(\overline{g}\Phi - g\overline{\Phi}) \equiv \mathrm{i}\, n\,(g\overline{h} - \overline{g}h) w_N\overline{w}_N \left[D\Sigma_2 - C_1\Sigma_2'(g\overline{h} + \overline{g}h) w_N\overline{w}_N\right]\,, \tag{28}$$

$$2h_0\dot{h}_0 = \mathrm{i}\, n(\overline{h}\Psi - h\overline{\Psi}) \equiv -\,\mathrm{i}\, n\,(g\overline{h} - \overline{g}h)\left[D\Sigma_2 - C_1\Sigma_2'(g\overline{h} + \overline{g}h) w_N\overline{w}_N\right]\,, \tag{29}$$

$$2f_0\dot{f}_0 = \mathrm{i}\, n(\overline{f}\Theta - f\overline{\Theta}) \equiv 0 \tag{30}$$

with $D = g^2\overline{g}^2 - h^2\overline{h}^2\,(w_N\overline{w}_N)^2$. With the use of

$$g\overline{h} - \overline{g}h = 2\,\mathrm{i}\, g_0 h_0 \sin(\xi - \eta), \qquad g\overline{h} + \overline{g}h = 2g_0 h_0 \cos(\xi - \eta) \tag{31}$$

one gets

$$\dot{g}_0 = n h_0 w_N\overline{w}_N \left\{D\Sigma_2 \sin(\eta - \xi) - g_0 h_0 w_N\overline{w}_N C_1\Sigma_2' \sin[2(\eta - \xi)]\right\}, \tag{32}$$

$$\dot{h}_0 = -n g_0 \left\{D\Sigma_2 \sin(\eta - \xi) - g_0 h_0 w_N\overline{w}_N C_1\Sigma_2' \sin[2(\eta - \xi)]\right\}, \tag{33}$$

$$\dot{f}_0 = 0\,. \tag{34}$$

4. NUMERICAL ESTIMATES

To get the approximate numerical estimates of the constants of our solution by comparing it with the classical SMART97 solution we use the same techniques as in (Brumberg and Ivanova, 2007). In accordance with the SMART97 solution the initial (for the epoch J2000) values of the Euler angles and their derivatives are

$$
\begin{aligned}
\psi^{(0)} &= -0.00006\,78954\,609\,, & \dot\psi^{(0)} &= 0.70105\,4959\cdot 10^{-6}/d\,,\\
\theta^{(0)} &= 0.40906\,46190\,715\,, & \dot\theta^{(0)} &= -0.09606\,7366\cdot 10^{-6}/d\,,\\
\varphi^{(0)} &= 4.89489\,89303\,002\,, & \dot\varphi^{(0)} &= 6.30038\,8130/d
\end{aligned}
\tag{35}
$$

(the angles ψ , ω , φ of SMART97 correspond to our angles $-\psi$, $-\theta$, φ, respectively). Starting with these initial values and the value $n = -3.1501\,9368/d$ we get the initial values (for the epoch J2000) for

$$
g_0^{(0)} = -\sin\frac{\theta^{(0)}}{2} = -0.2031\,0924\,, \quad h_0^{(0)} = \frac{1}{\sqrt{w_N\overline{w}_N}}\cos\frac{\theta^{(0)}}{2} = 70.7495\,9950\,, \tag{36}
$$

$$
f_0^{(0)} = \frac{1}{2\Omega}\sqrt{\frac{\omega_1^2}{k_2} + \frac{\omega_2^2}{k_1}} = .0000\,0058 \tag{37}
$$

where ω_1 and ω_2 are the components of the vector of the Earth rotation angular velocity referred to rotating Earth–fixed coordinate system. Then we obtain the numerical values for the first integrals of the secular system of the Earth's rotation

$$
C_1 = 1.0000000000000, \qquad C_2 = .0000000000003, \qquad C_3 = .0000000201237 \tag{38}
$$

and the values for

$$
\Delta = 1.0000\,0022\,, \qquad \sigma = 1.0000\,0001\,, \qquad \chi = -.0065\,6895\,, \tag{39}
$$

$$
n(\Delta+\sigma) = -6.3003\,8810/d\,, \qquad n(\Delta-\sigma) = -.0000\,0067/d\,, \qquad n\chi = .0206\,9345. \tag{40}
$$

Our results in $\dot\varphi$ and $\dot\psi$ coincide with the SMART97 solution up to $10^{-7}/d$. Then

$$
\xi_0^{(0)} = -\frac{\psi^{(0)}+\varphi^{(0)}}{2} = -2.4474\,1552\,, \qquad \eta_0^{(0)} = \frac{\pi}{2} + \psi^{(0)} - \gamma = 2.5467\,0110 \tag{41}
$$

where $\gamma = -3.4233\,8818$ was obtained in (Brumberg and Ivanova, 2007).

$\dot\zeta_0$ is determined from (27) with taking into account (31) and (34) and admitting for the approximate estimates $F - \chi = 0$ in (27)

$$
\dot\zeta_0 = n\sqrt{w_N\overline{w}_N}\,\sin\theta\,\frac{\Delta-\sigma}{C_1\Sigma_1}\,\Sigma_2''\cos(\eta-\xi)\,. \tag{42}
$$

The initial value (for the epoch J2000) for $\zeta_0^{(0)}$ is determined by

$$
\zeta_0^{(0)} = -\frac{1}{2}n\sqrt{w_N\overline{w}_N}\,\frac{\Delta-\sigma}{C_1\Sigma_1}\,\Sigma_2''\left[\frac{\cos(\theta^{(0)}+\eta_0^{(0)}-\xi_0^{(0)})}{\dot\theta^{(0)}-n\Delta+n\sigma} + \frac{\cos(\theta^{(0)}-\eta_0^{(0)}+\xi_0^{(0)})}{\dot\theta^{(0)}+n\Delta-n\sigma}\right] = -.0313\,8315\,. \tag{43}
$$

Six constants $g_0^{(0)}$, $h_0^{(0)}$, $f_0^{(0)}$, $\xi_0^{(0)}$, $\eta_0^{(0)}$, $\zeta_0^{(0)}$ are arbitrary constants of the trigonometrical solution of the Earth's rotation secular system.

5. CONCLUSION

The technique of this paper allows to construct a general theory of motion and rotation of the solar system bodies.

6. REFERENCES

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