Concise Algorithms for Precession-Nutation

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Presentation outline

- Motivation
- Methods
- Examples

The full-accuracy models are BIG

		number of terms					
		1	t	t^2	t^3	t^4	t^5
angles method	$\bar{\gamma}$	1	1	1	1	1	1
	$\bar{\phi}$	1	1	1	1	1	1
	ψ	1321	38	1	1	1	1
	ε_A	1038	20	1	1	1	1
X,Y method	X	1307	254	37	5	2	1
	Y	963	278	31	6	2	1
both	$s + XY/2$	34	4	26	5	2	1

Capitaine & Wallace (2006)

The need for concise formulations

- Use of the full models is justified when the application demands it and/or computing resources are ample.

- But often a lower accuracy will suffice, and then the full models are an unnecessary overhead.

- In some applications, computing resources are so limited that the full models are unaffordable. Either a simplified model must be used, or some interpolation or look-up scheme.

- The IAU has recognized this need, adopting the 2000B nutation series (McCarthy & Luzum 2003) as a limited-accuracy lightweight alternative to 2000A.

Example applications of concise NPB models

- Satellite orbit predictions.
- Pulsar timing (e.g. TEMPO2 uses IAU 2000B).
- Pointing of telescopes and antennas (~1″).
- Prediction of occultations.
- Alternative a priori model for IERS?

The goal

The transformation from celestial (GCRS) to terrestrial (ITRS) coordinates can be written as:

$$\boldsymbol{v}_{\mathrm{ITRS}} = \mathbf{R}_{\mathrm{PM}} \times \mathbf{R}_3(\theta) \times \mathbf{R}_{\mathrm{NPB}} \times \boldsymbol{v}_{\mathrm{GCRS}}$$

where θ is the Earth rotation angle (i.e. the nutation-precession-bias matrix $\mathbf{R}_{\mathrm{NPB}}$ is CIO based).

We wish to devise formulations for $\mathbf{R}_{\mathrm{NPB}}$ that achieve different compromises between accuracy and computing costs.

A starting point

- There are several ways of forming $\mathbf{R}_{NPB}$ (see Capitaine & Wallace 2006) and abbreviated forms of any of these could be developed.
- But the method based on direct series for X, Y and s+XY/2 is particularly attractive:
  - Bias, precession and nutation have already been combined.
  - Each of the three series makes the same contribution to final accuracy (e.g. no sin ε factors to consider).
  - Other aspects (such as the matrix formulation) can be optimized individually.
  - Simple truncation of the series is likely to deliver a nearly optimal result, without resorting to least-squares fitting or harmonic analysis.

Relationship to IAU 2000B

IAU 2000B already exists: why go further?

- Our goal is a CIO based $\mathbf{R}_{NPB}$: IAU 2000B is a classical nutation series.
- We want a range of performance options, not just one.
- By regarding a concise model as a low-precision delivery of the IAU standard, we reduce the need to create and maintain separate standards for concise forms.
- But IAU 2000B provides a natural benchmark, and in our examples we shall adopt the same target interval, namely 1995-2050, to aid comparison.

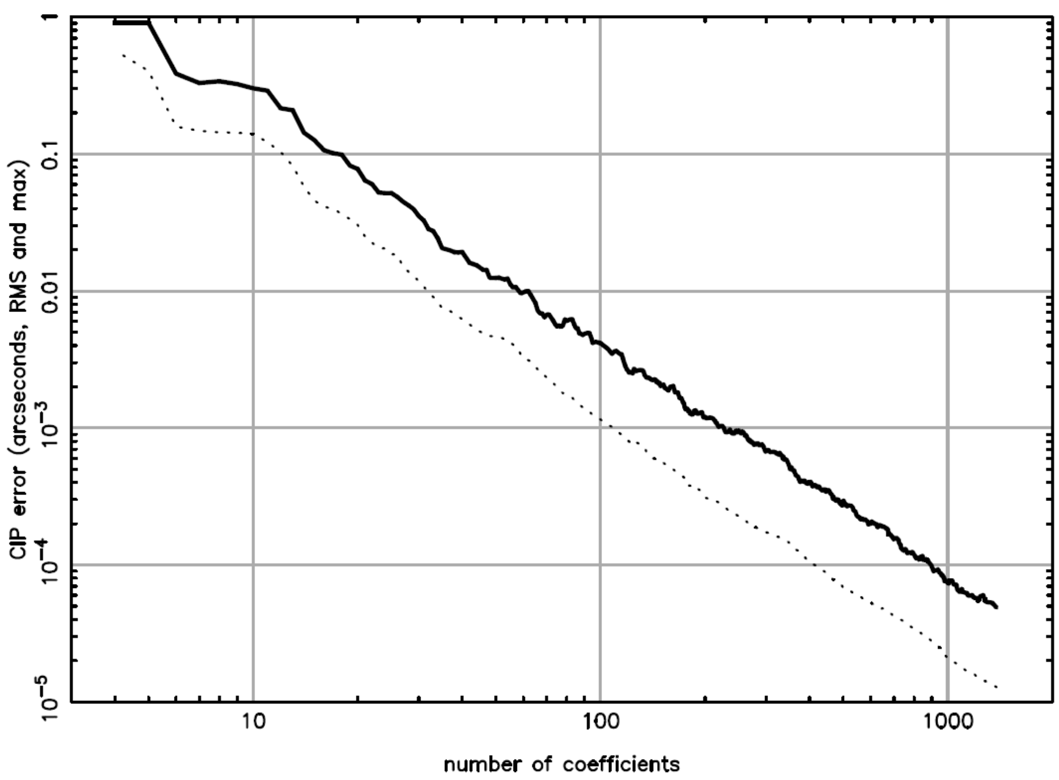
Opportunities for approximation

- The X, Y series.
- The s+XY/2 series.
- The matrix formulation (given X, Y and s).
- Fundamental-argument expressions.
- Long-period nutation.

X,Y series

- Truncating the X,Y series is where the biggest savings lie.
- Each term in X or Y consists of a sine and cosine component at a given frequency.
- The "purist" approach is to regard each term as a vector and to truncate based on modulus – so that each pair of coefficients is either cut or retained.
- But almost all terms have phases such that either the sine or cosine coefficient dominates.
- Truncating by individual coefficient, i.e. usually retaining only one of the pair, avoids ineffectual tiny values in the final series.

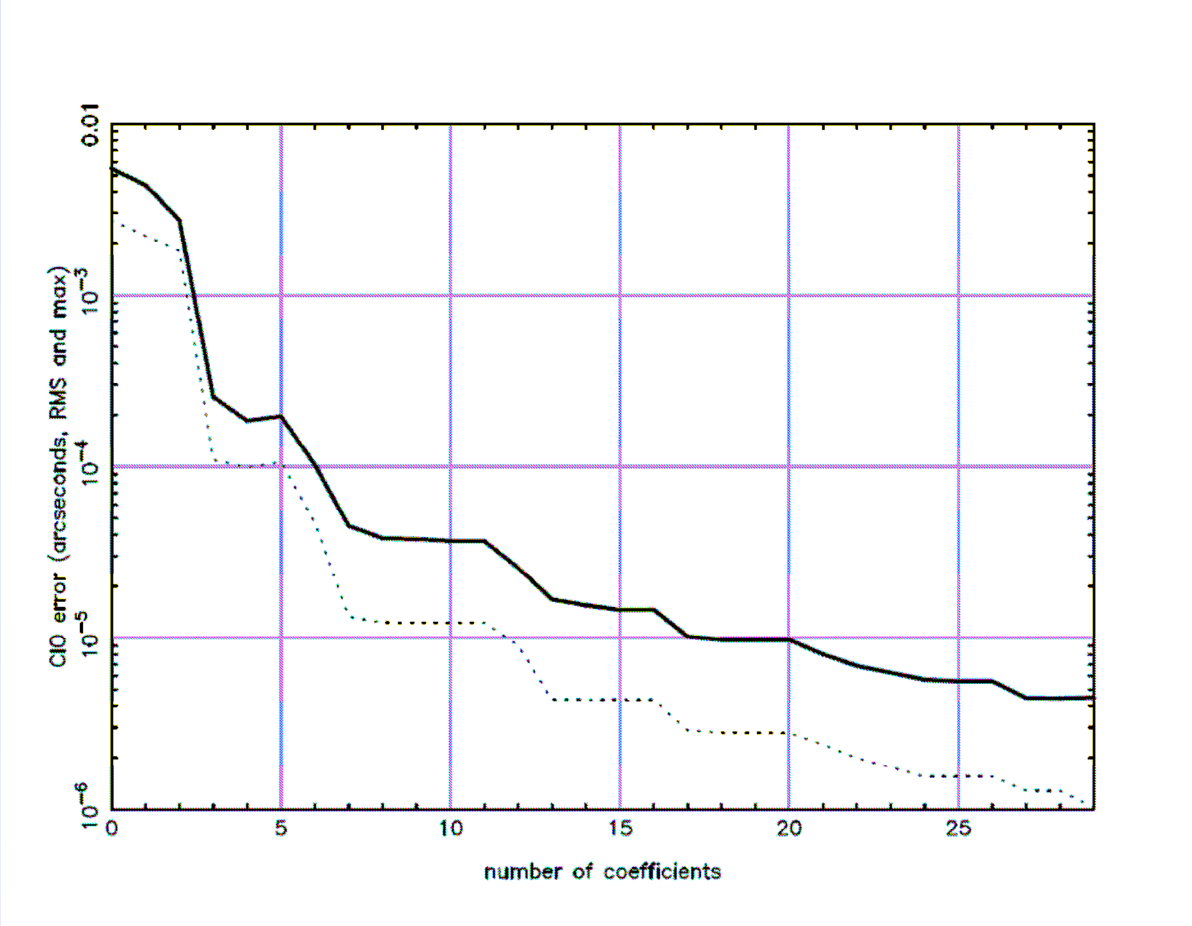
X,Y series: cut-off versus accuracy



s+XY/2 series

- The s+XY/2 series is much shorter than those for X and Y but there are still opportunities for worthwhile savings.
- Only a handful of terms is needed to achieve 1 mas.
- The X,Y values used to remove the XY/2 term do not have to be very accurate.
- For the most concise models, s can be neglected altogether.

S+XY/2 series: cut-off versus accuracy



R_{NPB} matrix formulation

$$\begin{pmatrix} \cos s + X(Y\sin s - X\cos s)/(1+Z) & -\sin s + Y(Y\sin s - X\cos s)/(1+Z) & -(X\cos s - Y\sin s) \\ \sin s - X(Y\cos s + X\sin s)/(1+Z) & \cos s - Y(Y\cos s + X\sin s)/(1+Z) & -(Y\cos s + X\sin s) \\ X & Y & Z \end{pmatrix}$$

rigorous

$$\begin{pmatrix} 1-\dfrac{X^2}{2} & -s-\dfrac{XY}{2} & -X \\ s-\dfrac{XY}{2} & 1-\dfrac{Y^2}{2} & -Y-sX \\ X & Y & 1-\dfrac{(X^2+Y^2)}{2} \end{pmatrix}$$

8 μas 2000-2100

165 μas 1800-2200

$$\begin{pmatrix} 1 & 0 & -X \\ 0 & 1 & -Y \\ X & Y & 1 \end{pmatrix}$$

0.12″ 2000-2100

0.85″ 1800-2200

The fundamental-argument expressions

- The X,Y and s+XY/2 series are functions of the *fundamental arguments*, a set of 14 angles.
- They comprise the five Delaunay variables l, l', F, D and Ω, eight planetary longitudes, and the general precession.
- Each is a polynomial in time: the expressions for the Delaunay variables use five coefficients (i.e. up to t^4) , all the others just two.
- Potential savings, from omitting unused arguments and truncating the series for the Delaunay variables, are always modest, but worthwhile for the more approximate $\mathbf{R}_{NPB}$ formulations.

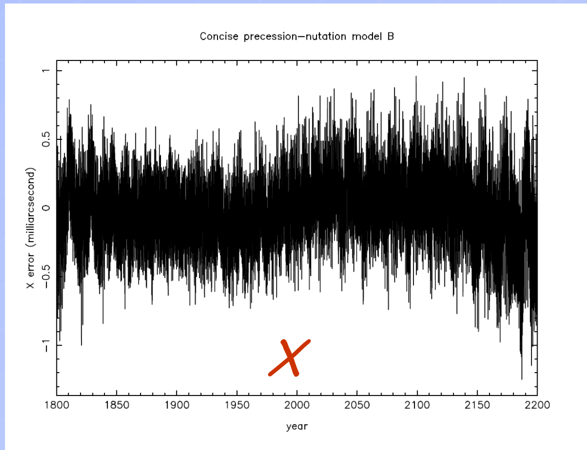
Long-period nutation

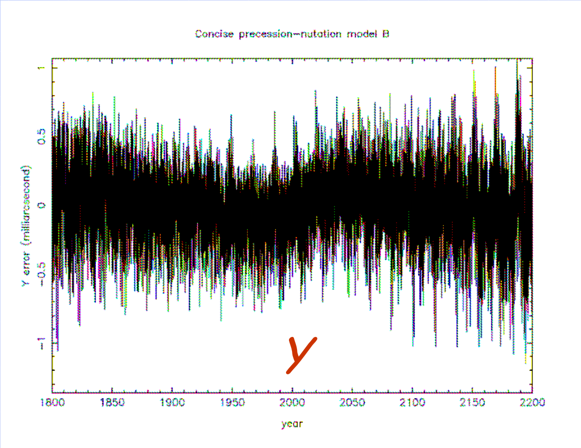
- The full X,Y series contain terms with periods from 3.5 days to 93.3 millennia.
- In a restricted time span (e.g. 1995-2050), the terms of longer period produce nearly fixed offsets in X and Y.
- Our examples use 1000 years as the cut-off, eliminating 33 terms and giving offsets of −634.2 μas in X and +1421.45 μas in Y.
- These are combined with the CIP bias and can then be used for all the concise formulations.

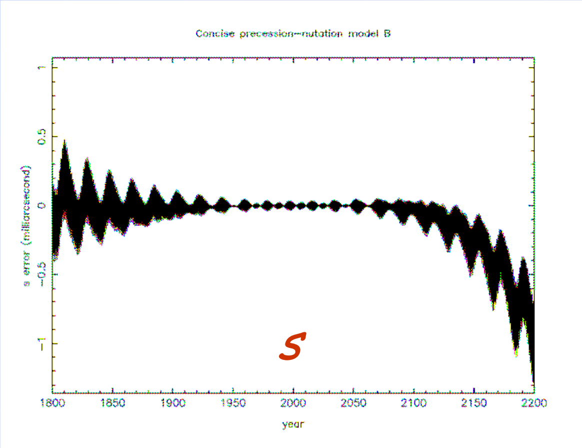
Example concise formulations CPN_b , CPN_c & CPN_d

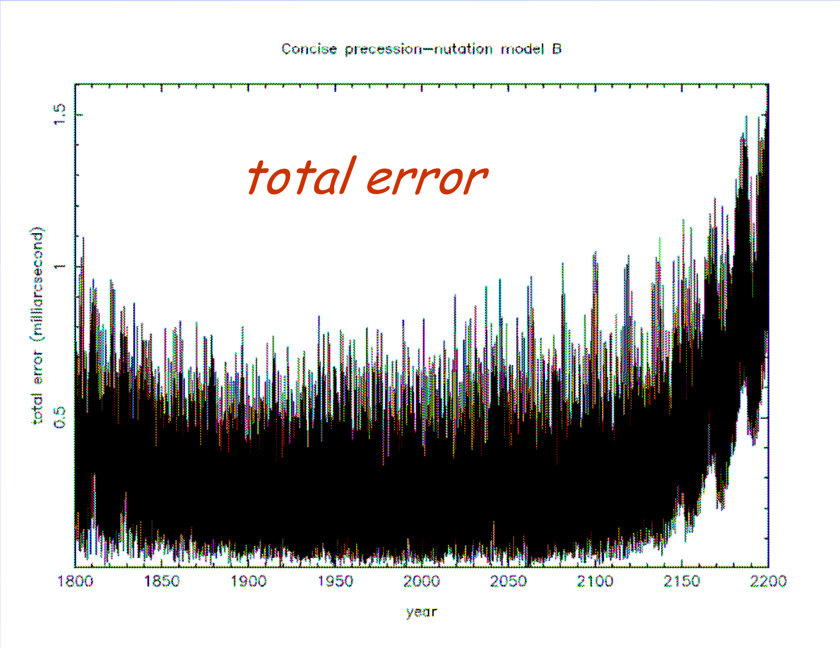
model	*coeffs*	*freqs*	*RMS*	*worst*	*speed*
reference	4006	1309	-	-	1
IAU 2000B	354	77	0.28	0.99	7.6
CPN_b	229	90	0.28	0.99	15.3
CPN_c	45	18	5.4	16.2	138
CPN_d	6	2	160	380	890
			mas	mas	

CPN_b : 400-year performance









- ~50 μas cut-off in X,Y, ~60 μas cut-off in s+XY/2.
- Better than 1 mas, 1995-2050.
- At the level where free core nutation is starting to matter.

Developments corresponding to CPN_b

$$
\begin{aligned}
X = \; & \xi_0 + \psi_A \sin\epsilon_0 - (\psi_A^3/6)\sin\epsilon_0 + \psi_A(\omega_A - \epsilon_0)\cos\epsilon_0 \\
& + \Delta\psi \sin\epsilon_0 + \Delta\psi\Delta\epsilon \; \cos\epsilon_0 \\
& (+\psi_A \cos\epsilon_0 - \chi_A) \;\; \Delta\epsilon + (\epsilon_A - \epsilon_0)\; \Delta\psi \cos\epsilon_0 \\
& -(\psi_A^2/2)\; \Delta\psi \sin\epsilon_0
\end{aligned}
$$

$$
\begin{aligned}
Y = \; & \eta_0 \;\; + (\omega_A - \epsilon_0) - (\psi_A^2/2)\sin\epsilon_0 \cos\epsilon_0 \\
& + (\psi_A^4/24)\sin\epsilon_0 \cos\epsilon_0 + d\alpha_0 \; \psi_A \sin\epsilon_0 \\
& + \Delta\epsilon - (\Delta\psi^2/2)\sin\epsilon_0 \cos\epsilon_0 \\
& -(\psi_A \cos\epsilon_0 - \chi_A)\; \Delta\psi \sin\epsilon_0 \\
& -(\psi_A^2/2)\cos\epsilon_0^2 \; \Delta\epsilon
\end{aligned}
$$

$$
\begin{aligned}
s + XY/2 = \; & [X_1 Y_0 + (1/2)\Sigma_i(a_i b_i \omega_i)] \; t + (1/3)\; X_1 Y_2 t^3 \\
& + (a_i b_i/4)\sin 2(\omega_i t - \phi_i) + (b_i/\omega_i) X_1 \sin(\omega_i t - \phi_i) \\
& + Y_2 t^2 \; a_i \sin(\omega_i t - \phi_i)
\end{aligned}
$$

Core terms (in b, c & d)

Further terms (in b & c)

Further terms (in b only)

CPN_c: 16 mas performance 1995-2050

term			amplitude	l	l'	F	D	Ω
X			−17251					
X	t		2004191898					
X	t^2		−429783					
X	t^3		−198618					
Y			−5530					
Y	t		−25896					
Y	t^2		−22407275					
$s+XY/2$	t		3809					
$s+XY/2$	t^3		−72574					
X		sin	−6844318	0	0	0	0	1
X	t	sin	−3310	"	"	"	"	"
X	t	cos	+205833	"	"	"	"	"
Y		cos	+9205236	"	"	"	"	"
Y	t	sin	+153042	"	"	"	"	"
$s+XY/2$		sin	−2641	"	"	"	"	"
X		sin	+82169	0	0	0	0	2
Y		cos	−89618	"	"	"	"	"
X		sin	+2521	0	0	0	2	0
X		sin	+5096	0	0	2	−2	1
Y		cos	−6918	"	"	"	"	"
X		sin	−523908	0	0	2	−2	2
X	t	cos	+12814	"	"	"	"	"
Y		cos	+573033	"	"	"	"	"
Y	t	sin	+11714	"	"	"	"	"
X		sin	−15407	0	0	2	0	1
Y		cos	+20070	"	"	"	"	"
X		sin	−90552	0	0	2	0	2
Y		cos	+97847	"	"	"	"	"
X		sin	−8585	0	1	−2	2	−2
Y		cos	−9593	"	"	"	"	"
X		sin	+58707	0	1	0	0	0
Y		cos	+7387	"	"	"	"	"
X		sin	−20558	0	1	2	−2	2
Y		cos	+22438	"	"	"	"	"
Y		cos	+2555	1	0	−2	−2	−2
X		sin	−4911	1	0	−2	0	−2
Y		cos	−5331	"	"	"	"	"
X		sin	−6245	1	0	0	−2	0
Y		cos	+3144	1	0	0	0	−1
X		sin	+28288	1	0	0	0	0
X		sin	+2512	1	0	0	0	1
Y		cos	−3324	"	"	"	"	"
Y		cos	+2636	1	0	2	0	1
X		sin	−11992	1	0	2	0	2
Y		cos	+12903	"	"	"	"	"

μas

- Coefficients can be rounded to 100 μas – though as a rule this will not affect compute time.
- 2-coefficient fundamental argument expressions are adequate for this formulation.

CPN_d: an extremely concise formulation

The expression:

$$\mathbf{R}_{NPB} = \begin{pmatrix} 1 & 0 & -X \\ 0 & 1 & -Y \\ X & Y & 1 \end{pmatrix}$$

where:

$$X = 0.00971660\,t - 0.00003318 \sin \Omega - 0.00000254 \sin A$$

$$Y = -0.00010863\,t^2 + 0.00004463 \cos \Omega + 0.00000278 \cos A$$

with t in centuries since J2000.0, and where:

$$\Omega = 2.182 - 33.757\,t \quad \text{radians}$$
$$A = -2.776 + 1256.664\,t \quad \text{radians}$$

Delivers **0.38 arcsecond accuracy throughout 1995-2050**, comparable with neglecting polar motion and adequate for pointing small telescopes (for example).

References

- Capitaine N., Wallace P.T., 2006, *High precision methods for locating the celestial intermediate pole and origin*, Astronomy & Astrophysics, **450**, 855-872
- McCarthy, D.D. & Luzum, B.J., 2003, *An abridged model of the precession-nutation of the celestial pole*, Celest. Mech. Dyn. Astron. **85**, 37-49

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