# VERIFICATIONS FOR MULTIPLE SOLUTIONS OF EARTH ROTATION

W.-J. WANG[1], W.-B. SHEN[2], H.-W. ZHANG[1]
[1] School of Surveying, Henan Polytechnic University, Jiaozuo, China
[2] School of Geodesy and Geomatics, Wuhan University, Wuhan, China
[1] email: wwj@asch.whigg.ac.cn, [2] email: wbshen@sgg.whu.edu.cn

ABSTRACT. In this study, we provide several approaches to the solution of the rotation of an arbitrary rigid body with triaxial principal moments of inertia. We found multiple solutions for the triaxial Earth rotation with two stable periodic trajectories and an unstable hyperbolic one. Data series filtering approach also shows explicit period of about 14 year in polar motion as well as in LOD.

## 1. EARTH'S SHAPE

The Earth has an anisotropic shape according to gravity or the geoid.

Table 1: Comparison of the polar and equatorial ellipticity for several planets

| | Earth | Mars | Moon | Hyperion | Eros433 |
|---|---|---|---|---|---|
| $H_0$ | $3.28475 \times 10^{-3}$ | $5.38 \times 10^{-3}$ | $5.2 \times 10^{-4}$ | 0.799 | 0.278? |
| $H_1$ | $2.1946 \times 10^{-5}$ | $5 \times 10^{-4}$ or $6.896 \times 10^{-4}$ | $5.5669 \times 10^{-5}$ | 0.158 | 0.158? |
| $e$ | 0.00382 | 0.06463 | 0.214 | 0.978 | 0.723 |

In this paper, we discuss the rotation of triaxial Earth starting from the triaxial rigid body; a discussion of pear-shaped Earth rotation can be found in Wang Wen-Jun [2004].

## 2. EXACT SOLUTION

Euler dynamical equations for rigid body rotation can be written equivalently as algebraic equations:

$$\begin{array}{lll} A\dot{\omega}_1 + (C-B)\omega_2\omega_3 = 0 & \dot{\omega}_1 + \sigma_1\omega_2\omega_3 = 0 & \sigma_2\omega_1^2 + \sigma_1\omega_2^2 = C_{12} \\ B\dot{\omega}_1 + (A-C)\omega_3\omega_1 = 0 \longrightarrow & \dot{\omega}_2 - \sigma_2\omega_3\omega_1 = 0 \longrightarrow & \sigma_3\omega_2^2 + \sigma_2\omega_3^2 = C_{23} \\ C\dot{\omega}_1 + (B-A)\omega_1\omega_2 = 0 & \dot{\omega}_3 + \sigma_3\omega_1\omega_2 = 0 & \sigma_1\omega_3^2 - \sigma_3\omega_1^2 = C_{31} \end{array}$$

where $C_{ij}$ are constants to be determined. The latter one gives two ellipses and a hyperbola. The solution of Euler equations is:

$$\omega_1 = \mu\sqrt{\frac{D(C-D)}{A(C-A)}}\mathrm{cn}u \qquad \omega_2 = -\mu\sqrt{\frac{D(C-D)}{B(C-B)}}\mathrm{sn}u \qquad \omega_3 = \mu\sqrt{\frac{D(D-B)}{C(C-B)}}\mathrm{dn}u$$

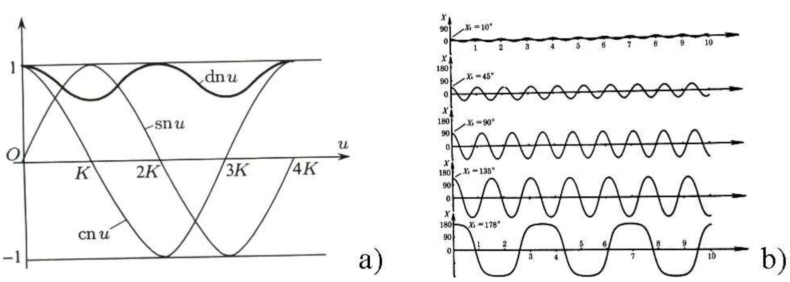


Figure 1: a) Graphs of elliptic functions b) cn$u$ compared with trigonometric $\cos u$

## 3. MOMENTUM CONSERVATION

Conservative momentum approach to the components of rotation angular velocity $\omega_i$, $i = 1, 2, 3$, results in the conservation for the angular momentum $H$ and for the potential energy $T \propto m^2$.

$$H = \frac{1}{2}M(\omega_1^2/A + \omega_2^2/B + \omega_3^2/C), \qquad \omega_1^2 + \omega_2^2 + \omega_3^2 = m^2 \qquad (1)$$

Let $x_i = \omega_i/m$, $(i = 1, 2, 3)$ and $\lambda_1 = 1/A$ , $\lambda_3 = 1/C$, $A < B < C$, $1/A > 1/B > 1/C$, here $1/C$ is the least. The first equation of (1) is changed to a quadratic polynomial of variables $x_i$ in which the respective coefficients $\lambda_i$ satisfy (+, +, +).

$$H = \frac{1}{2}m^2M(\lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3x_3^2) = \frac{1}{2}m^2M[(\lambda_1 - \lambda_3)x_1^2 + (\lambda_2 - \lambda_3)x_2^2 + \lambda_3] \qquad (2)$$

$$H = \frac{1}{2}m^2M[\lambda_1 + (\lambda_2 - \lambda_1)x_2^2 + (\lambda_3 - \lambda_1)x_3^2] \qquad (3)$$

Theory of quadratic polynomial tells us that if and only if the determinant of the last formula of (2) is positively definite, then it has stable elliptic solution, otherwise it has instable saddle point solution. For the case of $A \le B$ and $A \le C$, the quadratic polynomial (3) satisfies the positive definite condition (+, -, -). There must be another stable elliptic solution for the triaxial Earth rotation.

## 4. ALGEBRAIC CURVE EQUATIONS

Let $\sigma_1$, $\sigma_2$ and $\sigma_3$ be small quantities, defined by following equations

$$\alpha = \sigma_1 = (C - B)/A \quad \beta = \sigma_2 = (C - A)/B \quad \gamma = \sigma_3 = (B - A)/C \qquad (4)$$

## 5. EIGEN VALUE APPROACH

(4) allows us writing the Euler equations as vector form:

$$\begin{pmatrix} \dot{\omega_1} \\ \dot{\omega_2} \\ \dot{\omega_3} \end{pmatrix} = \begin{pmatrix} -\sigma_1\omega_2\omega_3 \\ \sigma_2\omega_3\omega_1 \\ -\sigma_3\omega_1\omega_2 \end{pmatrix} \longrightarrow \dot{\omega} = F(\omega)$$

with equilibrium for the nonlinear vector equation of Taylor expansion.

$$\dot{\omega} = F(\omega_0) + \dot{F}(\omega_0)(\omega - \omega_0) + O(\omega^2)$$

The derivative operator of the vector is obtained in Jacobian matrix.

$$\dot{F}(\omega) = \begin{pmatrix} 0 & -\sigma_1\omega_3 & -\sigma_1\omega_2 \\ \sigma_2\omega_3 & 0 & \sigma_2\omega_1 \\ -\sigma_3\omega_2 & -\sigma_3\omega_1 & 0 \end{pmatrix}$$

$$\dot{F}(\omega) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_2 \\ 0 & -\sigma_3 & 0 \end{pmatrix} \omega_1 + \begin{pmatrix} 0 & 0 & -\sigma_1 \\ 0 & 0 & 0 \\ -\sigma_3 & 0 & 0 \end{pmatrix} \omega_2 + \begin{pmatrix} 0 & -\sigma_1 & 0 \\ \sigma_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \omega_3$$

This provides decomposed theorem for triaxial rotation with each matrix describing the torque on a principal axis. By solving the eigen values of each matrix the behavior of the solution on each principal axis may be obtained. There are two stable solutions with elliptic orbits with the first and the third matrices for the minimum and the maximum axes of inertia as well as an unstable hyperbolic trajectory with the medium matrix appearing as a one-way inverted pendulum deemed as the secular wander in Goldreich & Toomre (JGR 1969: 10, 2555-2567).

## 6. CONCLUSION

According to the authors of this study, Earth rotation theory would need a new convention in which the two free wobbles should be added with similar consequences as Chandler wobble. The new convention may not be contradicted in the case of that of biaxial rotation.