TOWARDS THE RELATIVISTIC THEORY OF PRECESSION AND NUTATION

S.A. KLIONER, M.H. SOFFEL, Ch. LE PONCIN-LAFITTE
Lohrmann Observatory,
Dresden Technical University, 01062 Dresden, Germany

ABSTRACT. A numerical theory of Earth rotation has been constructed using the post-Newtonian model of rigidly rotating multipole moments. The theory is constructed by numerical integration and treats all spectrum of relativistic issues in a consistent way: (1) relativistic times scales, (2) relativistic scaling of astronomical constants, (3) relativistic torques and (4) geodetic precession as an additional torque in the equations of rotational motion with respect to the kinematically non-rotating GCRS. In the quasi-Newtonian limit our theory reproduces SMART97 within the accuracy of the latter.

1. RELATIVITY AND EARTH ROTATION

Earth rotation is the only astronomical phenomenon which is observed with very high accuracy, but modelled in a Newtonian way. Although a number attempts to estimate and calculate the relativistic effects in Earth rotation have been undertaken (e.g., Bizouard et al., 1992; Brumberg & Simon, 2007 and reference therein) no consistent theory has appeared until now. As a result the calculations of different authors substantially differ from each other. Even the way geodetic precession/nutation is usually taken into account is just a first-order approximation and is not fully consistent with relativity (see below). On the other hand, the relativistic effects in Earth rotation are relatively large. For example, the geodetic precession ($1.9''$ per century) is about 3×10^{-4} of general precession. The geodetic nutation (up to 200 μas) is 200 times larger than the goal accuracy of modern theories of Earth rotation. One more reason to carefully investigate relativistic effects in Earth rotation is the fact that the geodynamical observations give important tests of general relativity (e.g., the best estimate of the PPN γ using large range of angular distances from the Sun comes from geodetic VLBI data) and it is dangerous to risk that these tests are biased bacause of relativistically flawed theory of Earth rotation.

The main goal of the project, the first results of which are presented below, is to develop a theory of Earth rotation fully compatible with the post-Newtonian approximation of general relativity. A new consistent and improved precession/nutation theory has first to be developed for the model of rigidly rotating multipoles described in Klioner et al. (2001). Here we consequently use the post-Newtonian definitions of the potential coefficients, tensor of inertia, dynamical equations in the GCRS and relativistic time scales. Besides, for the first time we apply a rigorous treatment of the geodetic precession and nutation.

2. RELATIVISTIC EQUATIONS OF EARTH ROTATION

The model which is used in this investigation was discussed and published by Klioner et al. (2001). Not going into physical details of the model let us summarize the model (T is TCG here):

$$\begin{aligned}
\frac{d}{dT}\left(C^{ab}\,\omega^b\right) &= \sum_{l=1}^{\infty}\frac{1}{l!}\,\varepsilon_{abc}\,M_{bL}\,G_{cL} + \varepsilon_{abc}\,\Omega^b_{\rm iner}\,C^{cd}\,\omega^d, && (1)\\
C^{ab} &= P^{ac}\,P^{bd}\,\overline{C}^{cd}, \quad \overline{C}^{cd} = \text{const} && (2)\\
M_{a_1 a_2 \dots a_l} &= P^{a_1 b_1}\,P^{a_2 b_2}\dots P^{a_l b_l}\,\overline{M}_{b_1 b_2 \dots b_l}, \quad \overline{M}_{b_1 b_2 \dots b_l} = \text{const}, \quad l \ge 2, && (3)
\end{aligned}$$

where the angular velocity of the Earth is assumed to be related to the orthogonal matrix $P^{ab}(T)$ as

$$\omega^a = \frac{1}{2}\,\varepsilon_{abs}\,P^{db}(T)\,\frac{d}{dT}\,P^{dc}(T), \qquad (4)$$

and $\Omega^a_{\rm iner}(T)$ is the angular velocity of geodetic precession and nutation (e.g., Klioner et al. 2001). Here $\overline{C}^{cd}$ is the constant tensor of inertia of the Earth and $\overline{M}_{b_1 b_2 \dots b_l}$ is the constant multipole moments of gravitational field field of the Earth. The model given above assumes that both the tensor of inertia and the multipole moments rotate rigidly (that is, it is assumed that all the time-dependence of them is given by one and the same orthogonal matrix $P^{db}(T)$). The relativistic torque is described by a set of tidal multiple moments G_L that is given explicitly by Eqs. (19)–(23) of Klioner et al. (2001).

As discussed by Klioner et al. (2001) there is a number of assumptions in the model (1)–(4). These assumptions, being formal ones, lead to a system of differential equations (1) very similar to those of Newtonian rigid body as described, for example, by Bretagnon et al. (1997,1998). These assumptions will be relaxed at a later stage of the work which will be devoted to the effects of non-rigidity of the Earth.

3. RELATIVISTIC DEFINITIONS OF THE ANGLES

One of the tricky points is the relativistic definition of the angles describing the Earth orientation. Exactly as Bretagnon et al. (1997,1998) we first define the rotated BCRS coordinates (x, y, z) by two constant rotations of the BCRS as realized by the JPL's DE403:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_x(23^\circ 26' 21.40928'')\, R_z(-0.05294'') \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\rm DE403} \tag{5}$$

Then the IAU 2000 transformations between BCRS and GCRS are applied to the coordinates (t, x, y, z), t being TCB, to get the corresponding spatially rotated GCRS coordinates (T, X, Y, Z). The spatial coordinates (X, Y, Z) are then rotated to get the spatial coordinates of the terrestrial reference system (ξ, η, ζ):

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = R_z(\phi)\, R_x(\omega)\, R_z(\psi) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{6}$$

Angles ϕ, ψ and ω are then used to parametrize the orthogonal matrix P^{ab} and therefore, to define the orientation of the Earth orientation in the GCRS. The meaning of the terrestrial system (ξ, η, ζ) here is the same as in Bretagnon et al. (1997): this is the reference system in which we define the harmonic expansion of the gravitational field with standard values of the coefficients C_{lm} and S_{lm}.

4. OVERVIEW OF THE NUMERICAL CODE

A numerical integration code has been written in Fortran 95. The software is carefully coded to avoid numerical instabilities and excessive round-off errors. Two numerical integrators with dense output – ODEX and Adams-Bashforth-Moulton multistep integrator – are built into the code. These two integrators can be used to control each other. The integrations are automatically performed in two directions – forth and back – that allows one to directly estimate the accuracy of integration. The code is able to use any type of arithmetic available with a given current hardware and compiler. For a number of operations, which have been identified as precision-critical, one has the possibility to use either the library FMLIB (Smith, 2001) for arbitrary-precision arithmetic or an ad hoc code using two double-precision numbers to implement quadrupole-precision arithmetic. The Fortran code for operations with STF tensors have been automatically generated by a specially designed computer-algebraic program in *Mathematica*. Our current baseline is to use ODEX with 80 bit arithmetic. The estimated errors of numerical integrations after 150 years of integration are below 0.001 μas. The performance of the code in this set-up is about 7 seconds for one year of integration on a typical personal computer.

Several relativistic issues have been incorporated into the code: (1) the full post-Newtonian torque using the STF tensor machinery, (2) rigorous treatment of geodetic precession/nutation as an additional torque in the equations of motion, (3) rigorous treatment of time scales (any of four time scales - TT, TDB, TCB or TCG - can be used as the independent variable of the equations of motion (although TT is certainly preferable from the physical point of view), (4) correct relativistic scaling of constants and parameters (e.g., the mass parameter GM of the Earth compatible with TT is not compatible with TDB). In order to test our code and the STF-tensor formulation of the torque we have coded also the

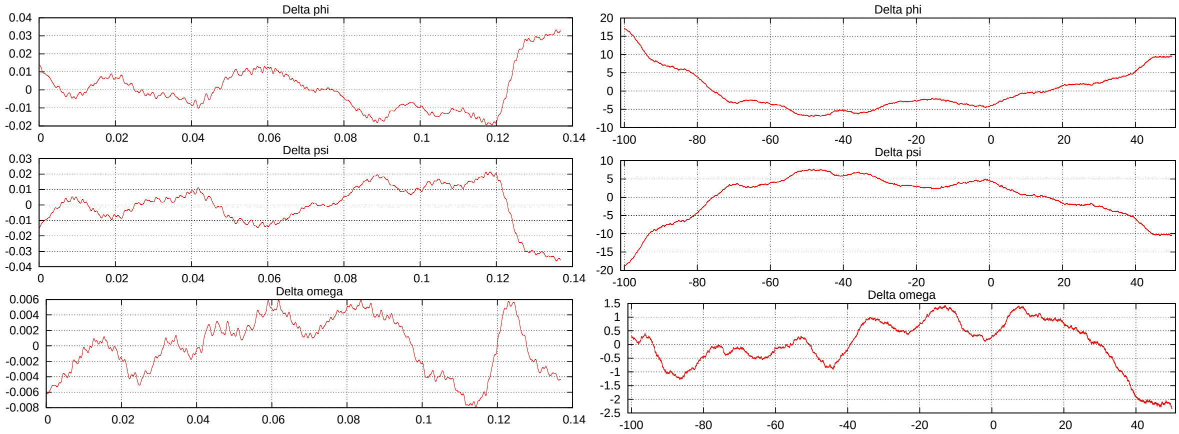

Figure 1: The differences in μas between the three angles ϕ, ψ and ω resulted from our numerical integrations and the full series of SMART97 for 50 days (left pane) and 150 years (right pane). The horizontal axes give time from J2000 in years. The curves on these plot closely repeat those on Figs. 3 and 4 of Bretagnon et al. (1998).

classical Newtonian torque with Legendre polynomials as described by Bretagnon et al. (1997, 1998) and integrated our equations for 150 years with these two torque algorithms. Maximal deviations between these two integrations was 0.0004 μas for ϕ, 0.0001 μas for ψ, and 0.0002 μas for ω. This demonstrates both the equivalence of the two formulations and the correctness of our code.

5. REPRODUCTION OF SMART97

As the first step, we have repeated the Newtonian dynamical solution of SMART97 using the Newtonian torque, the JPL ephemeris DE403 and the same initial values as in Bretagnon et al. (1997, 1998). Jean-Louis Simon (2007) has provided us with the unpublished full version of SMART97 (involving about 70000 Poisson terms for each of the three angles). We have calculated the differences between that full SMART97 series and our numerical integration. Detrended differences are shown on Fig. 1. A comparison of these differences and those given on Figs. 3–4 of Bretagnon et al. (1998) shows that we have succeeded to repeat SMART97 within the full accuracy of the latter.

6. THE EFFECT OF THE POST-NEWTONIAN TORQUE

In order to investigate the influence of the post-Newtonian torque in Eq. (1) as compared to the usual Newtonian torque, we have integrated the equations with the post-Newtonian and Newtonian torques and compared the results. The same initial conditions that we used to reconstruct the SMART97 solution were used for both integrations. Our results show that the post-Newtonian torque influences both precession and nutation. The effects in nutation are very small: the main effect is a correction to the main 18.6 year nutation term with the amplitude of 0.6 μas for ϕ and ψ and 0.4 μas for ω. This is very close to the estimates given by Bizouard et al. (1992). Besides that, secular or long-periodic terms appear in ϕ and ψ. A fit for the 150 years of integrations gives

$$\Delta\phi = 0.46 + 1492.85\,t - 8008.63\,t^2 + \dots \quad (7)$$

$$\Delta\psi = -0.49 - 1560.82\,t - 5.53\,t^2 + \dots \quad (8)$$

$$\Delta\omega = -0.22 - 0.31\,t - 6.22\,t^2 + \dots\ . \quad (9)$$

Here the angles are given in μas and t in thousand years from J2000. It is clear that on the interval of only 150 years it is not possible to distinguish between long-periodic and secular terms. Longer integrations for the full range of JPL's DE404 showed that, as one can expect, the effect is a mixture of long-periodic and polynomial terms. The linear drift of $\Delta\psi$ given above is, however, close to that appearing for 6000 years of integration. Details will be published elsewhere.

7. GEODETIC PRECESSION/NUTATION AS AN ADDITIONAL TORQUE

In the framework of our model (1)–(4) geodetic precession and nutation are taken into account in a natural way by including the additional torque proportional to $\Omega^a_{\rm iner}$ in the equations of rotational motion. This additional torque reflects the fact that the GCRS of the IAU is defined to be kinematically non-rotating (see, Soffel et al. 2003). This way to account for geodetic precession is more consistent than the way used previously by a number of authors: (1) solving the Newtonian equations of rotation as if these were the equations in the dynamically non-rotating version of the GCRS and (2) adding the precomputed geodetic precession/nutation. The second step is fully correct since the geodetic precession/nutation is by definition the rotation between the kinematically and dynamically non-rotating versions of the GCRS and it can be precomputed since it is fully independent of the Earth rotation. The inconsistency of the first step comes from the fact that in the computation of the Newtonian torque the coordinates of the solar system bodies are taken from an ephemeris constructed in the BCRS. However, the dynamically non-rotating version of the GCRS *rotates* relative to the BCRS with the angular velocity $\Omega^a_{\rm iner}(T)$. It means that the BCRS coordinates of solar system bodies should be first rotated into “dynamically non-rotating coordinates” and only after that rotation those coordinates can be used to compute the Newtonian torque. Loosely speaking one can say that in order to have better consistency the arguments of the Newtonian solution (e.g., the dynamical solution of SMART97) should be corrected for geodetic precession/nutation. It is, however, clear that it is much more consistent and much simpler to consider the geodetic effects as an additional torque as we do in this work. Comparison of the results of numerical integrations with the geodetic torque and the kinematical solution of SMART97 shows the differences that are in well agreement with the scheme depicted above and amount to up to 200 μas after 100 years of integration.

8. FURTHER STEPS

The preliminary results of our project have to be improved in several directions. First, the model of rigidly rotating multipoles has to be completed. We have to (1) clarify the meaning of the initial conditions for the angles and their derivatives in the relativistic context (this implies a clear distinction between quantities defined in kinematically and dynamically non-rotating coordinates), (2) find meaningful values of the moments of inertia $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in the relativistic context. After that the non-rigidity of the Earth should be treated.

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