ANALYTICAL THEORY FOR THE MOTION OF AN ASTEROID IN THE GRAVITATIONAL FIELD OF A MIGRATING PLANET

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ABSTRACT. A new perturbation method for the determination of proper elements of an asteroid in the gravitational field of a migrating planet is developed.

1. MIGRATION OF THE BIG PLANETS AND PERTURBATION THEORY

Planetary migration which, as is now widely accepted, had happened in the late stages of planetary formation (Fernadez & Ip 1984; Malhotra,1993), has forced the orbital resonances to sweep through a large region of the Solar system. Extremely time-consuming numerical simulations applied to a specifically formulated N-body problem (thousands of planetosimals with non-zero masses interact with the planets but not with each other) had strengthened the idea that some extraordinary orbital properties of Kuiper belt objects (including Pluto) are just a natural consequence of the migration of the big planets. On the other side, to follow the dynamical evolution of a concrete asteroid, it is sufficient to model the orbital migration of a perturbing planet with a time variation of its semi-major axis already gained from numerical simulations as was proposed in (Malhotra,1995):

$$a_b(t) = a_b^c - \Delta a_b \exp(-t/\tau)\,.$$

Here, a_b^c is the semi-major axis of the migrating planet at the current epoch, Δa_b is the migration interval over the total migration time τ. Being very simple, the implification of this formulae in classical perturbation techniques is really a challenge. First, the effect of migration introduces a non-standard time-dependence in the equations of the massless bodies; second, this effect is very small and acts over very long time spans which requires new standards for the orders of approximations and degrees of small parameters involved in the algorithm. It is also clear, that out of the two standard ways - successful integrations or canonical transformations - one should choose the second one in order not to suppress the migration effects by the artificial secular and mixed terms appearing in the first approach.

We have developed a special modification of the standard Lie-series based averaging procedure allowing to construct an analytical solution to the problem with an arbitrary migration law $a_b(t)$ in the absence of resonances, extendable to all successive approximations and to all degrees of eccentricities e, e_b and inclinations i, i_b.

To keep all the advantages of the Lie-series method, the scheme requires a canonical set of variables, for example, Delaunay elements (the final results have been transformed to non-singular elements) to keep the canonical form of the equations of motion. The chain of transformations induced by generating functions $W^{(k)}$ designed to eliminate successfully the angular variables transforms the initial system with Hamiltonian R into the systems with a new Hamiltonians $R^{(k)}$ with final averaged Hamiltonian depending only upon the action variables.

2. ELIMINATION OF MEAN ANOMALIES IN THE CASE OF A MIGRATING PLANET

In case of one perturbing planet migrating according to a prescribed low, the Hamiltonian R depends explicitly upon t only via the elements of the perturbing body L_b and $l_b = \int \bar{n}_b\, dt + l_b^0$ and expanding $W^{(1)}$ and $R^{(1)}$ in series of both the "natural" small parameter $\mu_b = \mathcal{M}_b/\mathcal{M}_{\rm Sun}$ and the small parameter induced by the slow migration rate $\varepsilon_b = \dot{a}_b/(a_b\, \bar{n} b)$, the solution can be found in a successive way through

$$\left(-\bar{n}\frac{\partial}{\partial l} - \bar{n}_b\frac{\partial}{\partial l_b} - \right) W_{j,0}^{(1)} + r_j = R_j^{(1)}, \quad \left(-\bar{n}\frac{\partial}{\partial l} - \bar{n}_b\frac{\partial}{\partial l_b}\right) W_{j,k+1}^{(1)} - \dot{a}_b\frac{\partial}{\partial a_b} W_{j,k}^{(1)} = 0,$$

allowing to find all the components of $W_{j,k}^{(1)}$ after having chosen the parts of the averaged Hamiltonian $R_j^{(1)}$. The first component $W_{j,0}$ corresponds to the standard solution of the Hamilton-Jacobi equation without migration.

3. ELIMINATION OF LONGITUDES OF PERIHELION AND NODE IN THE CASE OF A MIGRATING PLANET

After elimination of mean anomalies from the equations of asteroid motion, as new secular part of the lowest order Hamiltonian, supporting the elimination of long-periodic terms, we may only choose

$$R_0^{(1)} = -K + \mu_b \, (A \, e^2 + B \, s^2)$$

with coefficients A and B depending upon the semi-major axes, and K being conjugate to t. Even in the case without migration, one is confronted with some series problems. First, the rest of the Hamiltonian has μ_b as coefficient; this implies that this small parameter cannot support the further transformations and as new ones we are forced to consider the eccentricities and inclinations of the bodies involved. Every term of the form $e^{k1} \, s^{k2} \, e_b^{k3} \, s_b^{k4}$ should now be considered as a term of order $k = k1+k2+k3+k4$ in a small parameter ε representing e, e_b, s, s_b. This crucial point distinguishing the perturbation theory methods in the gravitational N-body problem from those of satellite problems has really been underestimated in celestial mechanics. Second, the Poisson bracket (a core operation of the algorithm) involves the derivatives w.r.t. e and $s = \sin i/2$, and that means, that with every approximation some (but not all) terms will loose the degree in small parameter. As a consequence, to get a solution correct up to a certain degree in ε , a different number of approximations are needed for different terms. Third, the part of the new Hamiltonian of second order results not only from secular terms, but also from some periodical terms which have to be considered separately. These second-order terms present in the case without migration a simplified case of Lagrange‘s secular solution which is principally limited to terms of second order, not only in the secular part of the perturbing function, but also in the Lagrange equations. Looking for an algorithm extendable to any desired degree and keeping in mind, that the Lagrange secular solution doesn't exist in case of migrating planets, we have developed an explicit algorithm based completely on the Lie-transform method. The solution has been realized with two simple transformations subsequently eliminating the longitudes of node and perihelia. For the first transformation, one may consider as secular part $R_0^{(1)} = \mu_b \, B \, s^2$, and then choose the according generating function as being dependent only upon h to hence remove the longitude of node from the problem. The elimination of the longitude of perihelion will be then realized with the choice $R_0^{(1)} = \mu_b \, A \, e^2$ (Tupikova, 2007). Due to the presence of D'Alembert characteristics, both of these two transformations converge formally, but some terms keep the same order of magnitude for a number of approximations (while not reducing their order) and then increase their order with step 2. To be sure that the solution has been obtained rigorously up to a given order, we have to divide all the terms in the development of the perturbing function into special "order groups" and then trace the magnitude of every term during the successive approximations. The migration effects have been treated in the way discussed above.

4. PRECISION OF THE ALGORITHM

To check the precision of the algorithm, we have chosen some "bad" terms in the Kaula-type expansion of R that converge extremely slowly and eliminated all the periodical perturbations up to 8th order. As a result, we have computed analytically the "proper" elements of an asteroid for 10^9 years from the values of the "osculating" elements gained from the numerical integration of the differential equations considering only these "bad terms". In spite of all the theoretical difficulties involved, in the absence of resonances the proper elements keep remarkably constant values.

5. REFERENCES

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