CONSIDERATIONS ABOUT SOME PROBLEMS ON FUNCTIONAL PARAMETRICAL MODELS IMPLEMENTATION FROM A DISCRETE SET OF DATA

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ABSTRACT. The least squares method is widely used in Fundamental Astronomy in the determination of some parameters that are usually coefficients of functional developments based on certain regularity hypothesis about the developed function. This hypothesis of regularity in the working domain, together with the spatial distribution of the discrete data and their statistical properties should be carefully treated if we want to obtain reliable results. The use of a kernel based method is shown as a robust procedure and allows to generalize, in numerical terms, the usual least-squares statistical treatment.

1. PROBLEMS INVOLVING PARAMETRICAL ADJUSTMENT FOR BIASED DATA

Given α_i, δ_i, Z_i be a set of n discrete values on the celestial sphere. A parametrical adjustment is given by the search of such c_k that $Z_i = \sum\limits_k c_k \Phi_k(\alpha_i, \delta_i)$ for functions Φ_k. Some aspects of the problem are: The points α_i, δ_i, are homogeneously distributed on the celestial sphere (other case can be similarly considered using an estimation of the density function from the discrete data), Z_i are the random variable values of Z. Z is also the function, searched as an adjustment, this will require some hypothesis (usually, the supposition of a integrable square on the sphere) The discrete distribution of Z_i values is determined by its mean and variance. The residual $R = Z - \sum\limits_k c_k \Phi_k$ is also understood as random variable as well as a function. With two random variables Z_i and Y_j over the same set α_i, δ_i, the search of the $Z_i = \sum\limits_k c_k \Phi_k(\alpha_i, \delta_i)$, $Y_i = \sum\limits_k d_k \Psi_k(\alpha_i, \delta_i)$ developments can be done with the same considerations that before. Another possibility is to suppose $c_k = d_k$ for a given reordenation of the Φ_k, Ψ_k basis. This case does not exclude the individual studies, but the basis functions will be different in each case. We shall use pairs $\Delta\alpha cos\delta$, $\Delta\delta$ or $\Delta\mu_\alpha cos\delta$, $\Delta\mu_\delta$ where Δ represents the differences in α, δ or in the proper motions $\mu_\alpha cos\delta$, μ_δ for two different catalogues. Provided the usual regularity hypothesis, there are spherical harmonics developments for each of the individual variables-functions. Analogously, it is possible to suppose a priori that the variables are related in pairs by means of infinitesimal rotations or spins, respectively. We use the same notation for the random variables and the function. The existence of infinitesimal rotations (or spins) should be compatible with the analytical and statistical properties for each variable-function considered. We are going now to deal with the functional and the statistical adjustment. They both are related to the most general least squares method: Given a function Z, which is defined in the unity sphere and has integrable square, then it may be developed in spherical harmonics. In practice, the series should be truncated up to an order m through the minimization problem $\left\| Z - \sum\limits_{k=0}^{m} c_k \Phi_k \right\|_2^2 = \frac{1}{4\pi} \int_{S^2} \left[Z - \sum\limits_{k=0}^{m} c_k \Phi_k \right]^2 d\sigma$, where the inner product is the usual in $L^2(S^2)$, and $c_k = (Z, Y_k)_2/(Y_k, Y_k)_2$ is $c_k = (Z, Y_k)_2$ in case of normalized basis. To notice the analogies with the statistical treatment, let us take the development in order zero whose coefficient is $c_0 = \int_{S^2} Z d\sigma$ which coincides with the mathematical expectation of Z (as a random variable). The harmonics spherical adjustment requires as a necessary condition the computation of the mean of the random variable that minimizes its variance, because c_0 minimizes $\frac{1}{4\pi} \int_{S^2} [Z - c_0]^2 d\sigma = Var(Z)$. In other words: the least squares method is the one that purposes, among all the unbiased estimators, the one with least variance (Gauss-Markov Theorem). If we identify Z and $\sum\limits_{k=0}^{n} c_k Y_k$ with their random variables, it is clear that $E[Z] = E\left[\sum\limits_{k=0}^{n} c_k Y_k \right] = c_0$, in consequence, the adjustment function preserves the mathematical expectation and minimizes the variance (among the estimators resulting from the truncation up to m-order). The natural generalization leads to the use of kernel non-parametrical models to obtain the regression of a random variable. In addition, it is also possible to consider a spatial distribution not necessary homogeneous, because the density functions are estimated by kernels. We are going to consider two problems involving parametrical adjustment for biased data. Problem 1: Let us take a lineal geometrical (GL) adjustment given by: $\Delta\alpha\cos\delta = \varepsilon_x\Phi_x + \varepsilon_y\Phi_y + \varepsilon_z\Phi_z$, $\Delta\delta = \varepsilon_x\Psi_x + \varepsilon_y\Psi_y$ then, $E[\Phi_x] = E[\Phi_y] = 0$, $E[\Phi_z] \neq 0$, $E[\Psi_x] = E[\Psi_y] = 0$ and $E[\Delta\alpha\cos\delta] = \varepsilon_z E[\Phi_z]$, $E[\Delta\delta] = 0$. To compute ε aiming to the second members to be estimators of least variance for the first, having into account that $r = \Delta\alpha\cos\delta - \sum\varepsilon_x\Phi_x$ (or $r = (\Delta\alpha\cos\delta)^2 + (\Delta\delta)^2$), the variance is $Var(r) = E[r^2] - (E[r])^2 = E\left[(\Delta\alpha\cos\delta - \sum\varepsilon_x\Phi_x)^2\right] - (E[\Delta\alpha\cos\delta] - \varepsilon_z E[\Phi_z])^2$ which has an extra term. The normal equations can not be the same, due to the existence of bias. The easiest way to solve this is the introduction of an auxiliary term in Right Ascension.

Problem 2: if the $\Delta\delta$ sample has bias, then the usual GL can not be used (if the sample is homogeneously distributed, then the computed values for the $E[\Psi]$ will be null). So, in practice, it is necessary to include an artificial term for this bias in order to generalize the GL model. The artificial terms included are null if there is no bias. Conclusion: if a random variable has bias, the adjustment should have this fact into account and it is absurd to consider an unbiased model with biased data. Several authors have not taken this fact into account and they have used unbiased models to adjust biased samples. The conclusions that they have reached are necessary wrong. Specially dealing with the rotations Hipparcos-FK5, the corrections of the parameters for the Luni-solar precession and the (fictitious) motion of the Equinox.

2. AUXILIARY USE OF THE NON PARAMETRICAL ADJUSTMENT FOR THE COMPUTATION OF THE PARAMETERS

We shall consider a one-dimensional distribution X. Its generalization in the sphere may be seen in (Marco et al., 2004, A&A 418) Let $x_1, ..., x_n$ be points, and $K(x)$ a function that is no negative in $[-1, 1]$, null in the rest and with integral the unity. Let $h_x > 0$, we define $f_h(x) = \frac{1}{nh_x}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h_x}\right)$ as an estimation of the density of the random variable X. (About the criteria to choose h_x, see for example Simonoff, 1996, "Smoothing Methods in Statistics", Springer Verlag). Consider a bidimensional distribution $(x_1, y_1), ..., (x_n, y_n)$. A regression function, may be defined as $m(x) = E[Y|X = x] = \int_D y\frac{f(x,y)}{f(x)}dy$. If we approximate the joint and marginal densities by means of K, then we obtain: $m(x) = \frac{1}{nh_x}\sum_{i=1}^{n} y_i K\left(\frac{x-x_i}{h_x}\right)$. Let us consider the minimization of $\frac{1}{nh_x}\int_D (y - m)^2 f_h(x)dx$ where $m = m(x)$ is the expectation of Y conditioned to $X = x$. The optimum m provides the previous expression; also, when obtaining m, the variance minimized is that of Y. So, it is natural the application of a non parametrical adjustment with the same properties than the parametrical (no bias for the definition of m, minimal variance due to the employed method). There is an extra advantage: we have a function defined over the whole domain (the celestial sphere) which preserves the statistical of the discrete data without making hypothesis about the geometrical properties of the function to be adjusted. If, for example, we denote as Ξ the (continuous) non parametrical adjustment for the $\Delta\mu_\alpha cos\delta$ data and we suppose that the function to adjust can be developed in spherical harmonics, then the coefficient of the normalized harmonics Y_i, is computed as the inner product (Ξ, Y_i) which should be discretised. Other important case is the computation of the rotations or the spins. We denote as Π the non parametrical adjustment for $\Delta\mu_\delta$, their bias is computed as $(\Pi, 1)$ and this value is independent of any model. This procedure has been used in the non parametrical adjustment for $\Delta\mu_\alpha cos\delta$ and $\Delta\mu_\delta$, and we have searched spins plus bias adjustment, through the conjunct minimization of $(\Delta\mu_\alpha cos\delta - Model)^2 + (\Delta\mu_\delta - Model)^2$. The obtained values for the bias coincide with the initial sample and the inducted values inducted for the spins unbiased by the bias and the biased spins coincide with the values obtained through products with spherical harmonics We conclude the existence of a total compatibility with the amplified geometrical model when there is bias. This is the case of the Luni-Solar precession and the fictitious Equinox motion. The consideration of bias in the model explains the discrepancies in the values of precession and Equinox motion much better than the models that do not consider bias.

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