GEOPOTENTIAL OF A TRIAXIAL EARTH WITH A RIGID INNER CORE IN ANDOYER CANONICAL VARIABLES

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ABSTRACT. We derive analytical expressions for the geopotential of a three-layer Earth model composed by a triaxial rigid mantle, a triaxial fluid outer core and a triaxial rigid inner core, extending previous works performed on the basis of an axial-symmetric three-layer Earth model (Greiner-Mai *et al.* 2000, 2001). In order to consider these expressions within the framework of the Hamiltonian theory of the rotation of the non-rigid Earth, we work out the problem in terms of a set of canonical variables arising from associating an Andoyer-like variables to each layer of the Earth, in the same way as it is described in Escapa *et al.*(2001). With the help of Wigner theorem (Kinoshita *et al.* 1974), we obtain the development of the geopotential of this Earth model in a mantle attached reference frame. Finally, we analyze the dependence of each geopotential coefficient of the second degree on the triaxiality and figure axis of the rigid inner core.

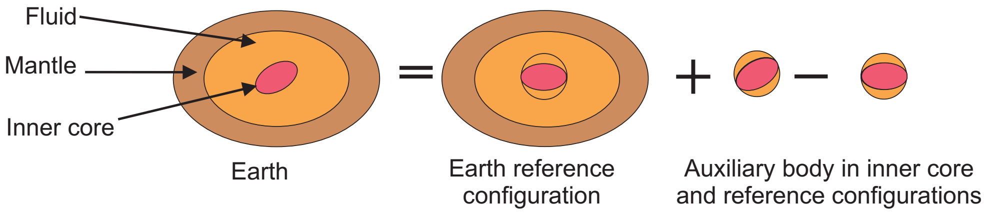
1. CANONICAL CONSTRUCTION OF THE GEOPOTENTIAL VARIATION

Let us consider a three-layer Earth model composed of a rigid mantle, an homogeneous fluid core and a rigid inner core, sharing a common barycenter O. Let us attach to the mantle and the inner core two reference frames $T_m = \{O; \vec{e}_{1m}, \vec{e}_{2m}, \vec{e}_{3m}\}$ and $T_s = \{O; \vec{e}_{1s}, \vec{e}_{2s}, \vec{e}_{3s}\}$ whose vectorial basis are constituted by the principal axis of the mantle and of the inner core, respectively. Under the prescribed conditions, the relative motion of the inner core with respect to the mantle is given by a rigid rotation of the reference frame T_s with respect to the reference frame T_m around O. As a consequence of this internal motion the density distribution inside the Earth viewed from T_m changes with time, it induces a time variation in the gravitational field originated by the Earth, that is to say, in the geopotential. By considering the decomposition sketched in the figure (Escapa *et al.* 2001) we infer that the temporal variation of the geopotential is exclusively due to the auxiliary body (subscript ab) and can be written as

$$\Delta U = U_{ab}(r, \delta', \alpha') - U_{ab}(r, \delta, \alpha), \tag{1}$$

where r, δ, α are the distance, latitude and longitude of the external point with respect to the frame T_m and their primmed counterparts are referred to the frame T_s. This expression should be referred to a single reference frame, which usually coincides with the reference frame of the mantle T_m. Since the frame T_s can be brought to the frame T_m by means of a rotation and considering the usual expansion of the geopotential in terms of the spherical harmonics (e.g. Kinoshita *et al.* 1974), the problem that we have faced is equivalent to determine how are transformed the spherical harmonics by a rotation. The solution of this problem is given by Wigner theorem (e.g. Kinoshita *et al.* 1974) that states that under a counterclockwise rotation around the x or z axes, which brings the frame T_m into de frame T_s (primed variables), the spherical harmonics in both frames are linearly related, being the coefficients of the linear combination functions of the parameters that characterize the rotation between the frames T_m and T_s.

One possible choice to describe these parameters is to employ the Andoyer-type canonical variables described in Escapa *et al.* (2001) for modelling analytically the rotation of a three-layer Earth model. This set is composed by eighteen variables: λ, μ, ν, Λ, N, M for the total Earth, λ_f, μ_f, ν_f, Λ_f, N_f, M_f for the fluid and λ_s, μ_s, ν_s, Λ_s, N_s, M_s for the inner core. The meaning of these variables (Escapa *et al.* 2001), and the auxiliary angles I, σ, I_f, σ_f, I_s and σ_s, are related with the angular momentum of the Earth, the fluid and the inner core and their projections in different reference frames. In particular,



it can be shown that the rotation which brings the T_m frame into the T_s frame is given by a rotation matrix $\mathbf{R}(\lambda_s,\, I_s,\, \mu_s,\, \sigma_s,\, \nu_s)$.

2. DISCUSSION OF THE SECOND DEGREE TERMS

Accordingly to the above described procedure, we can obtain for the second degree terms of the geopotential the following expression

$$\Delta U^{(2)} = \frac{G}{r^3} A_{ab} \sum_{m=0}^{2} \left\{ \left[e_{ab} a_{2m}(t) + d_{ab} \tilde{a}_{2m}(t) \right] C_{2m}\left(\delta,\alpha\right) + \left[e_{ab} b_{2m}(t) + d_{ab} \tilde{b}_{2m}(t) \right] S_{2m}\left(\delta,\alpha\right) \right\}$$

where we have introduced the ellipticity $e_{ab} = (C_{ab} - A_{ab})/A_{ab}$ and triaxiality $d_{ab} = (B_{ab} - A_{ab})/A_{ab}$ parameters and $C_{2m}\left(\delta,\alpha\right)$, $S_{2m}\left(\delta,\alpha\right)$ represent the real surface spherical harmonics of second degree and order m. This formula, which is a cumbersome function of the variables $\lambda_s, I_s, \mu_s, \sigma_s, \nu_s$ through the coefficients $a_{2m}(t)$, $\tilde{a}_{2m}(t)$, $b_{2m}(t)$ and $\tilde{b}_{2m}(t)$, can be simplified by assuming that in the rotational motion of the inner core the reference frames T_m and T_s are almost coincident. In addition, we will consider that the vectors $\vec{e}_{\vec{L}_s}$ and $\vec{e}_{3s}$ also remain close. These conditions are commonly assumed in Earth rotation studies of three-layer models. In terms of the canonical variables these geometrical relationships are expressed by the fact that the angles I_s, σ_s and $\nu_s + \mu_s + \lambda_s$ are small.

Therefore, we can keep only first order terms in these angles in the geopotential variations, obtaining the non-vanishing expressions

$$\begin{aligned} a_{21}(t) &= \sigma_s \sin\nu_s + I_s \sin\left(\nu_s + \mu_s\right), b_{21}(t) = -\left[\sigma_s \cos\nu_s + I_s \cos\left(\nu_s + \mu_s\right)\right], \\ \tilde{b}_{21}(t) &= \sigma_s \cos\nu_s + I_s \cos\left(\nu_s + \mu_s\right), \tilde{b}_{22}(t) = -\left(\nu_s + \mu_s + \lambda_s\right)/2. \end{aligned} \qquad (2)$$

From a geometrical point of view, the right hand sides of the equations (2) are related with the x and y components of the vector $\vec{e}_{3s}$ in the frame T_m, and with the angle between the vectors $\vec{e}_{1m}$ and $\vec{e}_{1s}$. Let us underline that we have obtained two extra terms to the geopotential variation, $\tilde{b}_{21}(t)$ and $\tilde{b}_{22}(t)$, which are caused by the triaxiality. Moreover, the temporal variation associated to the harmonic $S_{22}\left(\delta,\alpha\right)$ is exclusively due to the triaxiality and vanishes, in our order of approximation, in the axial-symmetric case. In view of the slight triaxiality of the inner core, as well as its small moment of inertia when compared with that of the Earth, these terms are expected to be small. Anyway, they could play a role for other planets or natural satellites that have a more massive inner core. On the other hand, it would be interesting to provide the analytical expressions that determine the temporal evolution of the canonical parameters $\lambda_s(t), I_s(t), \mu_s(t), \sigma_s(t), \nu_s(t)$, and therefore of $a_{2m}(t)$, $\tilde{a}_{2m}(t)$, $b_{2m}(t)$ and $\tilde{b}_{2m}(t)$, within the framework of the current Earth rotation models. This work is in progress and will be presented in a forthcoming communication.

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3. REFERENCES

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