INFLUENCE OF THE MULTIPOLE MOMENTS OF A GIANT PLANET ON THE PROPAGATION OF LIGHT : APPLICATION TO GAIA

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ABSTRACT. Approved space astrometry missions, like GAIA and SIM, are aimed to measure positions and/or parallaxes of celestial objects with an accuracy of 1-10 microarcseconds (μas). At such a level of accuracy, it will be indispensable to take into account the influence of the multipole structure of the giant planets (mainly Jupiter and Saturn) on the gravitational light deflection. Using the Nordtvedt-Will parametrized post-Newtonian formalism, we present an algorithmic procedure enabling to determine this influence on a light ray connecting two points located at a finite distance.

1. INTRODUCTION

Two major space astrometry missions, GAIA and SIM, are planned to be launched in the next years. The accuracy in the measurements of positions and/or parallaxes of celestial objects is expected to attain a level of 1-10 μas. In this context, we have to describe precisely the propagation of light inside and outside Solar System in a fully relativistic framework. By the time of Hipparcos mission, it was sufficient to consider the light deflection due to a static spherically symmetric Sun. Now, at the level of the μas accuracy, it is necessary to take into account the masses of the planets, as well as the higher multipole moments of those among them which are the most massive ones (Jupiter, Saturn, Uranus and Neptune). Table 1 gives the order of magnitude of the different contributions to the bending of a light ray propagating in Solar System. It is seen that for Jupiter, e.g., the effects of the multipole moments J_2 and J_4 may amount to 240 μas and 10 μas for a grazing light ray, respectively. So these effects must be taken into account in GAIA mission.

To take into account these intricate effects, several studies have been performed in the last decade. The first general relativistic model of positional observations at the level of 1 μas in space was proposed by Klioner & Kopeikin (1992), where gravitating bodies are considered as mass monopoles moving with constant velocities. More recently, a complete analytical description of the light propagation in the field of arbitrarily moving spinning mass monopoles bodies has been found by Kopeikin & Schäfer (1999) and Kopeikin & Mashhoon (2002) in the first post-Minkowskian approximation. For treating the particular problem of the multipole structure of celestial bodies, Hellings (1986) recommended to use the post-Newtonian formulas for the light propagation in the field of a motionless body and to introduce the position of each gravitating

Body	δ_{pn}	δ_{J_2}	δ_{J_4}	δ_{J_6}	δ_R	δ_{ppn}
Sun.................	1.752×10^6	1	—	—	0.7	11
Mercury...........	83	—	—	—	—	—
Venus.............	493	—	—	—	—	—
Earth................	574	0.6	—	—	—	—
Moon...............	26	—	—	—	—	—
Mars...............	116	0.2	—	—	—	—
Jupiter............	16270	240	10	$\geq$0.1	0.2	—
Saturn............	5780	95	6	$\geq$0.1	—	—
Uranus...........	2080	8	—	—	—	—
Neptune..........	2533	10	—	—	—	—

Table 1: Gravitational bending of light rays in Solar System. Here δ_{pn} and δ_{ppn} are the post-Newtonian and the post-post-Newtonian effects due to the spherically symmetric field of the body, δ_{J_2}, δ_{J_4} and δ_{J_6} are the effects due to multipole moments J_2, J_4 and J_6, respectively, and δ_R is the gravitomagnetic deflection. Each effect is evaluated for a grazing light ray. Unit is μas.

body at the moment of closest approach of that body by the photon. Klioner & Kopeikin (1992) apply this recommendation to treat the influence of the quadrupole moment of giant planets. Moreover, a rigourous formalism for determining the light propagation in the gravitational field of an isolated axisymmetric body was developped by Kopeikin (1997). However, the procedures given in these works are based on the analytical solution of the geodesic equations and requires cumbersome calculations. For this reason, only the influence of the quadrupole moment seems to be workable by this method.

Quite recently, we have reconsidered the problem of propagation of light between two events located at a finite distance in general spacetime (Linet & Teyssandier (2002) and Le Poncin-Lafitte & *al.* 2004). First of all, we have established a direct relation between the travel time of a photon and the vector tangent to the null geodesic at the emission point and the reception point, respectively. This means that all theoretical problems related to the direction of light rays may be solved as soon as the time transfer functions are determined. In addition, we have developed a procedure enabling to calculate explicitly the time travel of a photon in a general post-Minkowskian expansion, at any order of approximation without integrating the geodesic equations, even if the gravitational field is not stationary. Applying these results, we outline here a general method for determining the influence of the mass multipole moments of a planet on a light ray within the post-Newtonian approximation.

In section 2, we show that the angle between two light rays as measured by an observer can be computed when the time transfer functions are known. Then, we give the expression of this angle up to the order $1/c^3$ in the framework of the post-Newtonian approximation. In section 3, we restrict our attention to the case of an isolated, axisymmetric body. We suppose that the contribution of spin multipole moments are negligible, so that the gravitational field may be considered as a static one. On these assumptions, we give the contributions of the mass multipole moments to the time transfer function and to the direction of a light ray.

In this paper, G is the Newtonian gravitational constant and c is the speed of light in a vacuum. The Lorentzian metric of spacetime is denoted by g. The signature adopted for g is $(+ - - -)$. We suppose that spacetime is covered by a global coordinate system $(x^\mu) = (x^0, \boldsymbol{x})$, where $x^0 = ct$, t being a time coordinate, and $\boldsymbol{x} = (x^i)$, the x^i being quasi-Cartesian coordinates. We assume that the curves of equation $x^i = \ const$ are timelike, which means that $g_{00} > 0$ anywhere. We employ the vector notation $\boldsymbol{a}$ in order to denote either $(a^1, a^2, a^3) = (a^i)$ or $(a_1, a_2, a_3) = (a_i)$. Considering two such quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ with, for instance $\boldsymbol{a} = (a^i)$, we use $\boldsymbol{a}.\boldsymbol{b}$ to denote $a^i b^i$ if $\boldsymbol{b} = (b^i)$ or $a^i b_i$ if $\boldsymbol{b} = (b_i)$ (the Einstein convention on repeated indices is used). The quantity $|\boldsymbol{a}|$ stands for the ordinary Euclidean norm of $\boldsymbol{a}$. In what follows, greek indices run from 0 to 3, and latin indices run from 1 to 3.

2. ASTROMETRIC ANGLE WITHIN THE POST-NEWTONIAN APPROXIMATION

Let Γ_1 and Γ_2 be two light rays emitted at point $x_{A_1} = (ct_{A_1}, \boldsymbol{x}_{A_1})$ and $x_{A_2} = (ct_{A_2}, \boldsymbol{x}_{A_2})$ respectively and simultaneously received by an observer B located at point $x_B = (ct_B, \boldsymbol{x}_B)$. Let u be the unit 4-velocity of this observer. Denote by $l^{(1)}$ and $l^{(2)}$ the vectors tangent at x_B to Γ_1 and Γ_2, respectively. Since $l^{(1)}$ and $l^{(2)}$ are null vectors, the angle ϕ between these rays as measured by the observer B is given by

$$\cos\phi = 1 - \left[\frac{l^{(1)}.l^{(2)}}{(u.l^{(1)})(u.l^{(2)})}\right]_B . \tag{1}$$

This formula holds in any gravitational field. Using the quasi-Cartesian coordinates system (x^α) introduced in section 1, Equation (1) may be explicitly written as

$$\cos\phi = 1 - \left[\frac{g^{00} + g^{0i}\left(\frac{l_i^{(1)}}{l_0^{(1)}} + \frac{l_i^{(2)}}{l_0^{(2)}}\right) + g^{ij}\frac{l_i^{(1)}}{l_0^{(1)}}\frac{l_j^{(2)}}{l_0^{(2)}}}{(u^0)^2\left(1 + \frac{v^i}{c}\frac{l_i^{(1)}}{l_0^{(1)}}\right)\left(1 + \frac{v^i}{c}\frac{l_i^{(2)}}{l_0^{(2)}}\right)}\right]_B , \tag{2}$$

where $v^i = dx^i/dt$ is the coordinate velocity of the observer. In Le Poncin-Lafitte & *al.* (2004), we showed that the ratio l_i/l_0 can be explicitly determined when the time transfer functions are known. Let us recall that in general the travel time $t_B - t_A$ of a photon between an emission point $(ct_A, \boldsymbol{x}_A)$ and a reception point $(ct_B, \boldsymbol{x}_B)$ may be considered as a function of t_A, $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, or as a function of t_B, $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, so that we can put

$$t_B - t_A = \mathcal{T}_e(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B) = \mathcal{T}_r(t_B, \boldsymbol{x}_A, \boldsymbol{x}_B) , \tag{3}$$

where $\mathcal{T}_e$ and $\mathcal{T}_r$ may be called the emission and reception time transfer functions, respectively. We proved in the above-mentioned paper that the ratio l_i/l_0 is given at reception point x_B by the relation

$$\left(\frac{l_i}{l_0}\right)_B = -c\frac{\partial\mathcal{T}_e}{\partial x_B^i} = -c\frac{\partial\mathcal{T}_r}{\partial x_B^i}\left[1 - \frac{\partial\mathcal{T}_r}{\partial t_B}\right]^{-1} . \tag{4}$$

In what follows, we use the post-Newtonian approximation, so that the metric tensor may be written as

$$g_{00} = 1 - \frac{2W}{c^2} + O\left(\frac{1}{c^4}\right) , \tag{5}$$

$$\{g_{0i}\} = \{h_{0i}\} = \vec{h} = O\left(\frac{1}{c^3}\right) , \tag{6}$$

$$g_{ij} = \left(1 + 2\gamma\frac{W}{c^2}\right)\eta_{ij} + O\left(\frac{1}{c^4}\right) , \tag{7}$$

where $W = U + O(1/c^2)$, U being the Newtonian-like potential of the body. For a light ray emitted at point x_A and received at point x_B, we may write

$$\frac{l_i}{l_0} = -N^i + \Delta_i , \tag{8}$$

where

$$N^i = \frac{x_B^i - x_A^i}{R_{AB}}, \quad R_{AB} = |\boldsymbol{x}_B - \boldsymbol{x}_A|$$

and Δ_i is the relativistic contribution to the light deflection. As a consequence, Eq. (2) becomes

$$\begin{aligned}\cos\phi &= 1 - \frac{1-\frac{v^2}{c^2}}{\left(1-\frac{\vec{v}}{c}.\boldsymbol{N}^{(1)}\right)\left(1-\frac{\vec{v}}{c}.\boldsymbol{N}^{(2)}\right)}\Big\{(1-\boldsymbol{N}^{(1)}.\boldsymbol{N}^{(2)})\Big[1-(\boldsymbol{N}^{(1)}+\boldsymbol{N}^{(2)}).\vec{h} \\ &\quad -\frac{v^i}{c}(\Delta_i^{(1)}+\Delta_i^{(2)})\Big] + [N^{(2)i} - (\boldsymbol{N}^{(1)}.\boldsymbol{N}^{(2)})N^{(1)i}]\Delta_i^{(1)} \\ &\quad +[N^{(1)i} - (\boldsymbol{N}^{(1)}.\boldsymbol{N}^{(2)})N^{(2)i}]\Delta_i^{(2)}\Big\} + 0\left(\frac{1}{c^4}\right), \qquad (9)\end{aligned}$$

where

$$N^{(1)i} = \frac{x_B^i - x_{A_1}^i}{|\boldsymbol{x}_B - \boldsymbol{x}_{A_1}|}, \quad N^{(2)i} = \frac{x_B^i - x_{A_2}^i}{|\boldsymbol{x}_B - \boldsymbol{x}_{A_2}|}, \qquad (10)$$

Let us note that Eq. (9) holds even if the gravitational field is not stationary.

3. TIME TRANSFER AND LIGHT DEFLECTION

Let us apply these results to a light ray propagating in the field of an isolated, axisymmetric body. We suppose that the gravitational effects of the internal angular momentum of the body may be neglected. So we consider that the gravitational field is static. The center of mass of the body being taken as the origin O of quasi-Cartesian coordinates $(\boldsymbol{x})$, we choose the axis of symmetry as the x^3-axis. We put $r = |\boldsymbol{x}|$, $r_A = |\boldsymbol{x}_A|$ and $r_B = |\boldsymbol{x}_A|$. We denote by $\boldsymbol{k}$ the unit vector along the x^3-axis and we consider only the case where all points of the segment joining $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ are outside the body. We denote by r_e the radius of the smallest sphere centered on O and containing the body (for celestial bodies, r_e is the equatorial radius). We assume the convergence of the multipole expansions formally derived below at any point outside the body, such that $r > r_e$. On the above-mentioned assumptions, the two time transfer functions $\mathcal{T}_e$ and $\mathcal{T}_r$ reduce to a single function $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ which may be expanded as

$$\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{c}R_{AB} + (\gamma+1)\frac{GM}{c^3}R_{AB}F(0, \boldsymbol{x}_A, \boldsymbol{x}_B) + \sum_{n=2}^{\infty}\mathcal{T}_{W,J_n}(\boldsymbol{x}_A, \boldsymbol{x}_B) + 0\left(\frac{1}{c^4}\right), \qquad (11)$$

where $F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B)$ is defined by

$$F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{R_{AB}}\ln\left(\frac{|\boldsymbol{x}-\boldsymbol{x}_A| + |\boldsymbol{x}-\boldsymbol{x}_B| + R_{AB}}{|\boldsymbol{x}-\boldsymbol{x}_A| + |\boldsymbol{x}-\boldsymbol{x}_B| - R_{AB}}\right), \qquad (12)$$

and $\mathcal{T}_{W,J_n}$ is the contribution of the mass multipole moment J_n, given by

$$\mathcal{T}_{W,J_n}(\boldsymbol{x}_A, \boldsymbol{x}_B) = -(\gamma+1)\frac{GM}{c^3}\frac{1}{n!}J_n r_e^n R_{AB}\frac{\partial^n}{\partial(x^3)^n}F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B))|_{\boldsymbol{x}=0}. \qquad (13)$$

Calculating explicitly the successive derivatives of $F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B)$, we find

$$\begin{aligned}\mathcal{T}_{W,J_n}(\boldsymbol{x}_A, \boldsymbol{x}_B) &= -(\gamma+1)\frac{GM}{c^3}J_n r_e^n\sum_{m=1}^{n}(-1)^m\frac{(m-1)!}{2^m} \\ &\quad \times\frac{(r_A+r_B+R_{AB})^m - (r_A+r_B-R_{AB})^m}{(r_A r_B + \boldsymbol{x}_A.\boldsymbol{x}_B)^m} \qquad (14) \\ &\quad \times\sum{}'\left\{\frac{1}{i_1!i_2!\ldots i_{n-m+1}!}\prod_{l=1}^{n-m+1}\left[\frac{1}{r_A^{l-1}}C_l^{(-1/2)}\left(\frac{\boldsymbol{k}.\boldsymbol{x}_A}{r_A}\right) + \frac{1}{r_B^{l-1}}C_l^{(-1/2)}\left(\frac{\boldsymbol{k}.\boldsymbol{x}_B}{r_B}\right)\right]^{i_l}\right\},\end{aligned}$$

where $C_l^{(-1/2)}$ denote the Gegenbauer polynomial of degree l with parameter $-1/2$ (see Abramowitz and Stegun 1970) and $\sum'$ is a summation over all positive integers $i_1, i_2, ..., i_{n-m+1}$, solutions to the linear system

$$i_1 + 2i_2 + ... + (n-m+1)i_{n-m+1} = n\,, \qquad i_1 + i_2 + ... + i_{n-m+1} = m\,.$$

We are now in a position to determine the covariant components of the vector tangent to the light ray emitted at point x_A and received at point x_B. Applying Eqs. (4), (11) and (14), and then noting that one may set $l_0 = 1$ along the ray since the gravitational field is static, we find at point x_B

$$(\boldsymbol{l})_B = -\boldsymbol{N} + (\boldsymbol{l}^W)_B\,, \tag{15}$$

where

$$(\boldsymbol{l}^W)_B = -(\gamma+1)\frac{GM}{c^2}\frac{(r_A+r_B)\boldsymbol{N} - R_{AB}\boldsymbol{n}_B}{r_A r_B + \boldsymbol{x}_A.\boldsymbol{x}_B} + \sum_{n=2}^{\infty}(\boldsymbol{l}^{W,J_n})_B(\boldsymbol{x}_A,\boldsymbol{x}_B)\,, \tag{16}$$

with

$$\begin{aligned}
\left(\boldsymbol{l}^{W,J_n}\right)_B(\boldsymbol{x}_A,\boldsymbol{x}_B) &= (\gamma+1)\frac{GM}{c^2}J_n r_e^n\left\{\sum_{m=1}^{n}(-1)^{m+1}m!\right.\\
&\times\left[\frac{\boldsymbol{n}_B-\boldsymbol{N}}{(r_A+r_B-R_{AB})^{m+1}} - \frac{\boldsymbol{n}_B+\boldsymbol{N}}{(r_A+r_B+R_{AB})^{m+1}}\right]\sum{}'\frac{1}{i_1!i_2!...i_{n-m+1}!}\prod_{l=1}^{n-m+1}D_{(l)}^{i_l}\\
&+\sum_{m=1}^{n}(-1)^m(m-1)!\left[\frac{1}{(r_A+r_B-R_{AB})^m} - \frac{1}{(r_A+r_B+R_{AB})^m}\right]\\
&\times\sum{}'\left[\frac{1}{i_1!i_2!...i_{n-m+1}!}\sum_{l=1}^{n-m+1}\frac{i_l}{r_B^l}D_{(l)}^{i_l-1}\prod_{p=1\,,\,p\neq l}^{n-m+1}D_{(p)}^{i_p}\right.\\
&\times\left.\left.\left[\boldsymbol{n}_B P_l\left(\frac{\boldsymbol{k}.\boldsymbol{x}_B}{r_B}\right) - \boldsymbol{k}P_{l-1}\left(\frac{\boldsymbol{k}.\boldsymbol{x}_B}{r_B}\right)\right]\right]\right\}\,,
\end{aligned} \tag{17}$$

P_l being the Legendre polynomial of degree l, and $D_{(l)}$ being defined by

$$D_{(l)} = \frac{1}{r_A^{l-1}}C_l^{(-1/2)}\left(\frac{\boldsymbol{k}.\boldsymbol{x}_A}{r_A}\right) + \frac{1}{r_B^{l-1}}C_l^{(-1/2)}\left(\frac{\boldsymbol{k}.\boldsymbol{x}_B}{r_B}\right)\,.$$

The contributions to angle ϕ due to $\boldsymbol{l}^{W,J_n}$ are on current study now.

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