THE LUNAR LIBRATION: COMPARISONS BETWEEN VARIOUS MODELS-A MODEL FITTED TO LLR OBSERVATIONS

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ABSTRACT. We consider 4 libration models : 3 numerical models built by JPL (ephemerides for the libration in DE_{245}, DE_{403} and DE_{405}) and an analytical model improved with numerical complements fitted to recent LLR observations. The analytical solution uses 3 angular variables (p_1, p_2, τ) which represent the deviations with respect to Cassini's laws. After having referred the models to a unique reference frame, we study the differences between the models which depend on gravitational and tidal parameters of the Moon, as well as amplitudes and frequencies of the free librations. It appears that the differences vary widely depending of the above quantities. They correspond to a few meters displacement on the lunar surface, reminding that LLR distances are precise to the centimeter level. Taking advantage of the lunar libration theory built by Moons (1984) and improved by Chapront *et al.* (1999a) we are able to establish 4 solutions and to represent their differences by Fourier series after a numerical substitution of the gravitational constants and free libration parameters. The results are confirmed by frequency analyses performed separately. Using DE_{245} as a basic reference ephemeris, we approximate the differences between the analytical and numerical models with Poisson series. The analytical solution - improved with numerical complements under the form of Poisson series - is valid over several centuries with an internal precision better than 5 centimeters.

1. PRESENTATION OF THE MODELS

• *Cassini's laws.* The lunar libration is characterized by small oscillations around an equilibrium position governed by Cassini's laws: *(i)*, the rotation period of the Moon is identical to its circulation period in the orbital motion; *(ii)*, the inclination of the lunar equator on the ecliptic is a constant; *(iii)*, the secular motions of the nodes N and N' on the ecliptic of the orbital plane and the lunar equator are identical (see Fig. 1).

• *The variables.* A selenodesic system of axes (ξ, η, ζ) along the principal moments of inertia (A, B, C) is connected with the ecliptic system (X, Y, Z) by 3 Euler's angles (ϕ, ψ, θ) as shown on Fig. 1. Three position angles denoted by (p_1, p_2, τ) express the small oscillations around the equilibrium position. They are referred to Euler's angles by the relation:

$p_1 = \sin\phi\sin\theta$; $p_2 = \cos\phi\sin\theta$; $\tau^* = \phi + \psi$ or $\tau = \tau^* - w_1 - 180°$

where w_1 is the mean longitude of the Moon. p_1 and p_2 are the components of the unit vector pointing towards the mean pole of the ecliptic of date on the two lunar equatorial principal axes of inertia; τ is the libration in longitude. In the analytical theory the position variables are (p_1, p_2, τ). In the 3 JPL lunar ephemerides Euler angles are used instead.

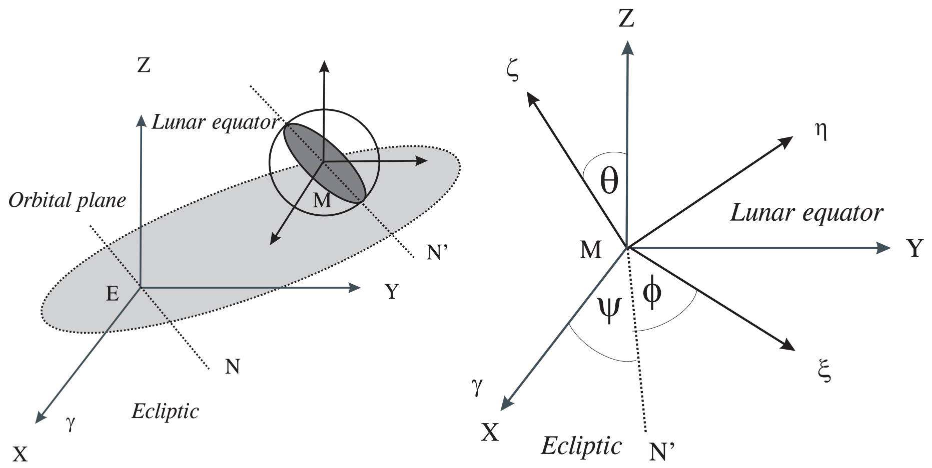

Figure 1: Ecliptic frame, selenodesic system of reference, and Euler's angles

Table 1: Parameters of the various models

Fundamental arguments	DE_{245}	DE_{403}	DE_{405}	$S_{LLR}(ICRS)$
ϕ	-0.07355	-0.05294	-0.05028	-0.05542
$\epsilon - 23°26'21"$	0.40580	0.4092	0.40960	0.41100
$w_1^{(0)} - 218°18'59"$	0.83482	0.87484	0.87267	0.8782
$w_1^{(1)} - 1732559343$ "$/Cy$	0.35614	0.35624	0.32953	0.3328
$w_1^{(2)}$ "$/Cy^2$	-6.7996	-6.7772	-6.8368	-6.8700
Earth figure parameters				
Units of 10^{-4}				
C_{30}	-0.086802	-0.086474	-0.087855	-0.086802
C_{31}	0.307083	0.307083	0.308038	0.307083
C_{32}	0.048737	0.048727	0.048798	0.048737
C_{33}	0.017161	0.017655	0.017702	0.017161
S_{31}	0.046115	0.044875	0.042593	0.046115
S_{32}	0.016975	0.016962	0.016955	0.016975
S_{33}	-0.002844	-0.002744	-0.002710	-0.002844
γ	2.278860	2.278642	2.278583	2.278860
β	6.316191	6.316107	6.316121	6.316191
$\frac{C}{mR^2}$	3948.723999	3950.296917	3952.951990	3948.723999
Free libration				
$\sqrt{2P}$	0.2933	0.2915	0.3047	0.2919
$\sqrt{2Q}$	5.1924	5.2095	5.1860	5.2722
$\sqrt{2R}$	0.0208	0.0218	0.0089	0.0217
p_0 *degree*	224.3029	224.3095	229.5037	224.2277
q_0 *degree*	161.6400	161.7655	160.9133	161.0477
r_0 *degree*	124.3936	109.6807	164.2453	98.7506
ω_p (observed) "$/Cy$	44820553.89	44820553.89	44819747.03	44820417
ω_q (observed) "$/Cy$	1736494.75	1736523.38	1736520.37	1736493.0
ω_r (computed) "$/Cy$	-5364715.23	-5364708.34	-5364600.06	-5364715.23

• *The libration theory.* The analytical theory of the lunar libration that we use is due to Moons (1981, 1982 and 1984). It contains 'forced libration' and 'free libration' series. All the components of the Moons' series are described in (Chapront *et al.*, 1999a) that include also further improvements due to the authors. The literal parameters which enter the 'forced libration' are: $\beta = \frac{C-A}{B}$, $\gamma = \frac{B-A}{C}$, where A, B and C are the lunar principal moments of inertia; and the ratios of the coefficients $C_{i,j}, S_{i,j}$, $(i = 3,4,\ 0 \leq j \leq i)$ to $\frac{C}{m_L R_L^2}$, where m_L and R_L are the lunar mass and equatorial radius. The corresponding Fourier series are developped with angular variables which are linear combinations of the 4 Delaunay's arguments D, F, l, l', the planetary mean longitudes λ_k $(1 \leq k \leq 8)$ and ζ, the lunar mean longitude referred to the mean equinox of date. The tidal perturbations introduce time-dependant analytical corrections $\Delta C_{i,j}$ and $\Delta S_{i,j}$ to the harmonics $C_{i,j}$ and $S_{i,j}$. The 'free libration' is described by 3 literal parameters $\sqrt{2P}$, $\sqrt{2Q}$, $\sqrt{2R}$ (constants of integration). In the corresponding Fourier series enter the 3 arguments of the free libration denoted by p, q, and r in addition to Delaunay's arguments.

• *Frame and constants.* A comparison between various JPL ephemerides of number n, i.e. $DE_n = DE_{245}, DE_{403}, DE_{405}$, and the analytical solution fitted to LLR observations, supposes that we use in all solutions the same set of constants and the same reference frame. Using the analytical theory ELP for the orbital motion of the Moon, by comparison with the JPL lunar ephemeris, we determine the reference frames of DE_n referred to the ecliptic and the inertial mean equinox for J2000.0, as well as the lunar mean longitude w_1 referred to a fixed equinox. The basic angles are: ϕ, separation between the origin of right ascensions of DE_n and the inertial equinox of J2000.0 along the equator of DE_n, and ϵ, the obliquity of DE_n (Chapront, *et. al.*, 1999b, 2002). Having brought the solutions in the same frame of reference, and using the sets of physical parameters listed in Table 1, a frequency analysis on the residuals between the analytical and the numerical models allows to evaluate the amplitudes of the 'free libration', i.e. the numerical values of literal parameters $\sqrt{2P}$, $\sqrt{2Q}$, $\sqrt{2R}$, as well as the libration frequencies: ω_s, $(s = p, q, r)$, with the following notations: $p = \omega_p t + p_0$, $q = \omega_q t + q_0$, $r = \omega_r t + r_0$; we call these frequencies 'observed values' in the sense that they are obtained by an ajustment to an ephemeris which represents the observations. Besides, the frequencies ω_s can be computed from their literal expressions provided by the theory; we call them 'computed values'.

2. COMPARISONS OF THE MODELS WITH DE_{245} AS REFERENCE

• *A semi-analytic form of the libration series.* On the basis of the analytical series, we have established 4 various solutions S_n $(S_{245}, S_{403}, S_{405}, S_{LLR})$ under the form of Fourier series with numerical coefficients. For each solution S_n, the coefficients have been computed with the numerical values given in Table 1. Hence, we have generated the series p_1^n, p_2^n and τ^n for any of the solutions S_n corresponding to the 3 JPL ephemerides DE_n and to the ephemeris E_{LLR} obtained by the authors with a LLR fit (see description hereafter). The quantities which are retained to adjust the solutions are ϕ, ϵ, $w_1 = w_1^{(0)} + w_1^{(1)} t + w_1^{(2)} t^2$, the free libration parameters (amplitude, phases and frequencies for p, q and r). In all the analytical series S_n, the arguments are linear combinations of Delaunay's arguments, planetary longitudes, ζ, and the angles p, q and r, whose frequencies are 'observed' through harmonic analysis as mentioned above. Nevertheless r cannot be determined with enough accuracy by harmonic analysis. We used the 'computed' value instead. The ephemeris E_{LLR} which corresponds to our fit to LLR observations is derived from DE_{245}. We have first established numerical complements to the analytical solution S_{245}, i.e. ρ_{245}, in order to get $S_{245} + \rho_{245} = DE_{245}$. Next, we have fitted the parameters to the LLR observations covering the period [1974-2002] (Chapront *et al.*, 2002). Using the values of the fitted paramers, we have obtained a new solutions S_{LLR} and a new ephemerides $E_{LLR} = S_{LLR} + \rho_{245}$.

• *A crude comparison of the models.* We choose as a reference DE_{245} which is also the model providing the Earth figure parameters in S_{LLR}. Formely, this solution has been a reference to elaborate the numerical complements to ELP and the libration ephemeris (Chapront *et al.*, 1997). We compute for the 3 variables p_1, p_2 and τ the differences: $\Delta E_n = DE_n - DE_{245}$ for DE_{403} and DE_{405}; in case of the LLR ephemerides, we form the difference: $\Delta E_{LLR} = E_{LLR} - DE_{245}$. Over a short time interval (1968-2010) we illustrate the differences on Fig. 2. For the variable τ, the graphs 'ΔE_{405}' and 'ΔE_{403}' have been shifted as follows: $\tau_{405} - \tau_{245} + 3".9$; $\tau_{403} - \tau_{245} + 2".2$. The differences ΔE_n reach maximum values as large as 0".3. That represents on the lunar surface a displacement of a few meters while the individual LLR observations have an accuracy at the centimeter level. In the process of comparisons and fits to observations, the scatter between the models is reflected on the determinations of the frames, the values of the physical parameters, the positions and velocities of the stations and reflectors.

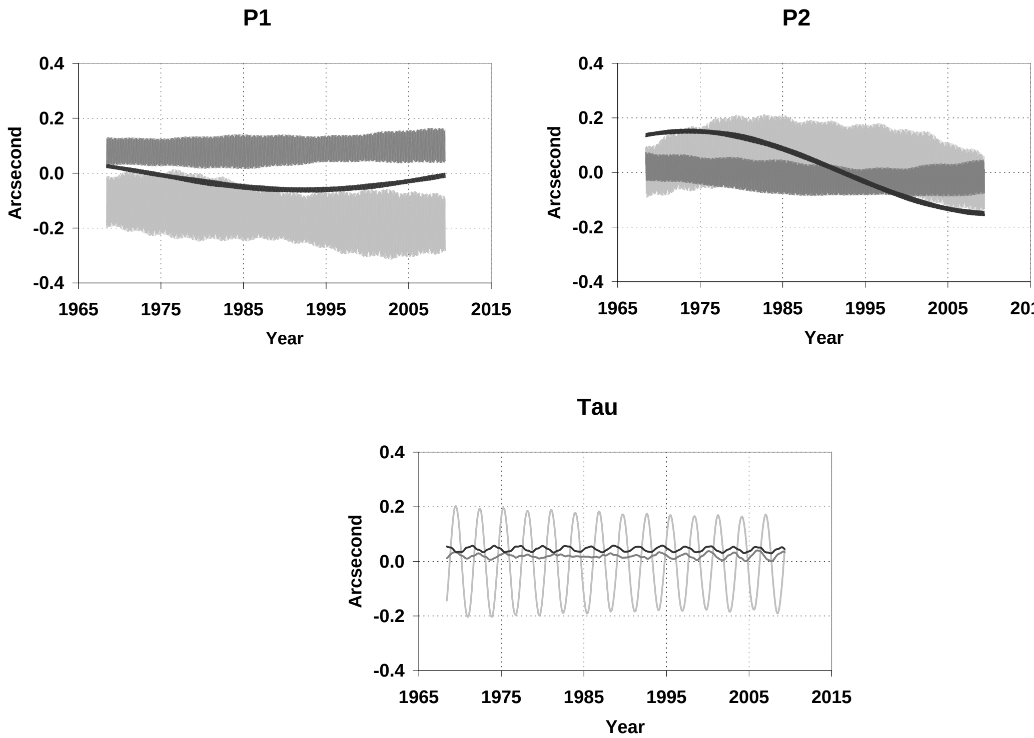

Figure 2: Differences of various models with respect to DE_{245}; (a) light grey: $\Delta E_{405} = DE_{405} - DE_{245}$; (b) dark grey: $\Delta E_{403} = DE_{403} - DE_{245}$; (c) black: $\Delta E_{LLR} = E_{LLR} - DE_{245}$

• *Approximation of the differences.* The solutions although they are far from each others, can be brought closer in a very simple manner with the aid of the analytical solution. In the case of DE_{405} (Standish, 1998), on one side we build the differences on the source ephemerides $\Delta E_{405} = DE_{405} - DE_{245}$; on the other side we build the differences on the analytical solutions $\Delta S_{405} = S_{405} - S_{245}$. An identical work can be performed with DE_{403} with results qualitatively very close. The residuals between ΔE_{405} and ΔS_{405} are shown on Fig. 3. They are explicitely described with short series of about ten terms. The extremum between the residuals are about 0".01 over 3 centuries. Using a sofware due to Mignard (2003), we have performed a frequency analysis of the differences ΔE_{405} and we have obtained ΔE^*_{405}. The related differences $\Delta E_{405} - \Delta E^*_{405}$, which are also represented on Fig. 3, are smaller than above ($\leq$ 0".005) which corresponds to the centimeter level. We note also that this approximation is valid on a very long time interval: [1750-2050]. It is worth noticing that we find in ΔE^*_{405} the main arguments (or frequencies) explicitely given in the analytical differences ΔS_{405}. For the variable τ, the deviation in $\Delta E_{405} - \Delta S_{405}$ is due to 2 periodic terms with close frequencies. Only one exists in S_{405}, the second one has been detected by harmonic analysis.

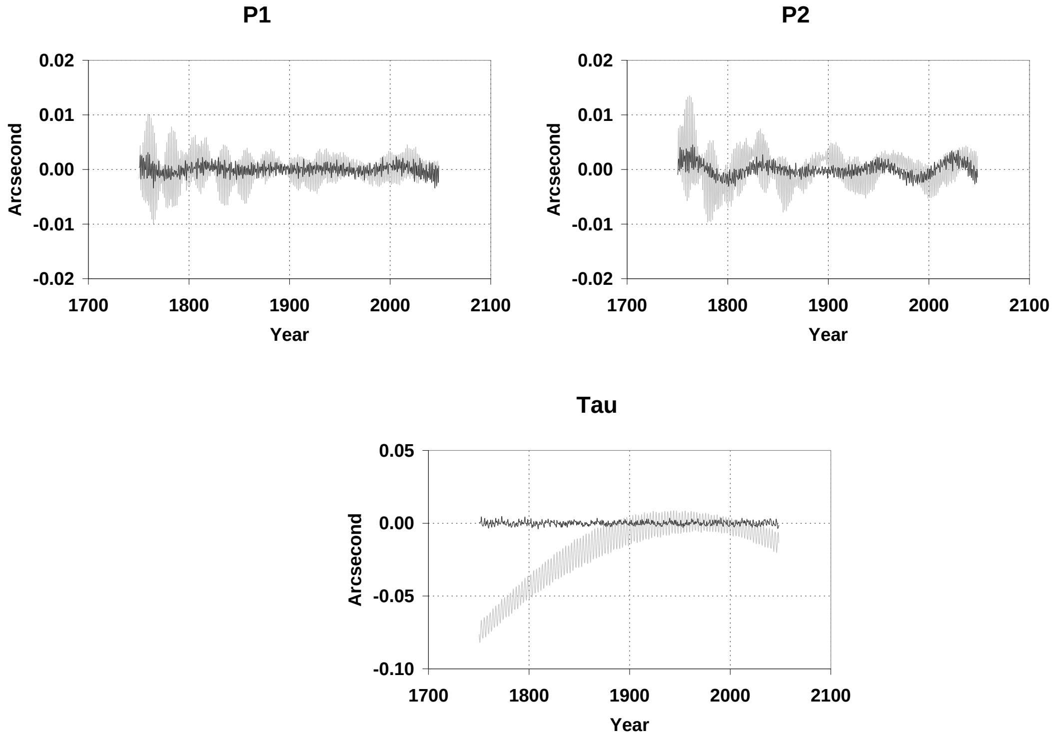

Figure 3: Comparison between numerical and analytical differences for DE_{405}; (a) light grey: Analytical solution, $\Delta E_{405} - \Delta S_{405}$ (b) black: Frequency analysis, $\Delta E_{405} - \Delta E^*_{405}$

3. PSEUDO-ANALYTICAL COMPLEMENTS WITH POISSON SERIES

Since we have at our disposal an analytical representation of the libration series (p_1, p_2, τ), and in particular a list of arguments (or frequencies) corresponding to the Fourier terms , we have completed and improved the analytical series by Poisson series. We have used a method which has been formely elaborated to improve, over a long time span, planetary analytical series (Chapront, 2000). We compute the differences between the numerical ephemeris and give to any variable σ the following form which is called P_n, or Poisson approximation of the difference $E_n - S_n$ related to the ephemeris $E_n = DE_n$ or E_{LLR}:

$$\sigma = \sigma_0 + \sigma_1 t + \sum_{[j]} \sum_{k} (C_0^{(k)} + C_1^{(k)} t) \cos(j_1\lambda_1 + j_2\lambda_2 + ...) + (S_0^{(k)} + S_1^{(k)} t) \sin(j_1\lambda_1 + j_2\lambda_2 + ...)$$

The numerical coefficients σ_s, $C_s^{(k)}$, $S_s^{(k)}$, $(s = 0, 1)$, are determined by least square fits. On the Fig. 4, in the case of DE_{245}, one represents the differences $E_n - S_n$ between numerical and analytical ephemerides and a comparison to its approximation P_n. The graph '$E_n - S_n$' shows the crude differences between the numerical and the analytical model. The graph '$E_n - (S_n + P_n)$' shows the small-sized residuals after the approximation of the analytical solution with Poisson series. The maximum of the differences is of the order of 0".01.

4. CONCLUSION

This study puts in evidence that it is possible to pass from a model of the lunar libration to another one, with the addition of short Fourier series. Besides, a chosen analytical solution S_n completed by Poisson terms represent the libration variables with a great accuracy over several centuries. Our final choice is the solution fitted to the LLR observations, S_{LRR}. A complete analytical model plus its pseudo-analytical complements, as well as a FORTRAN software to built an ephemeris of the libration variables, can be found on the website : http://syrte.obspm.fr/polac.

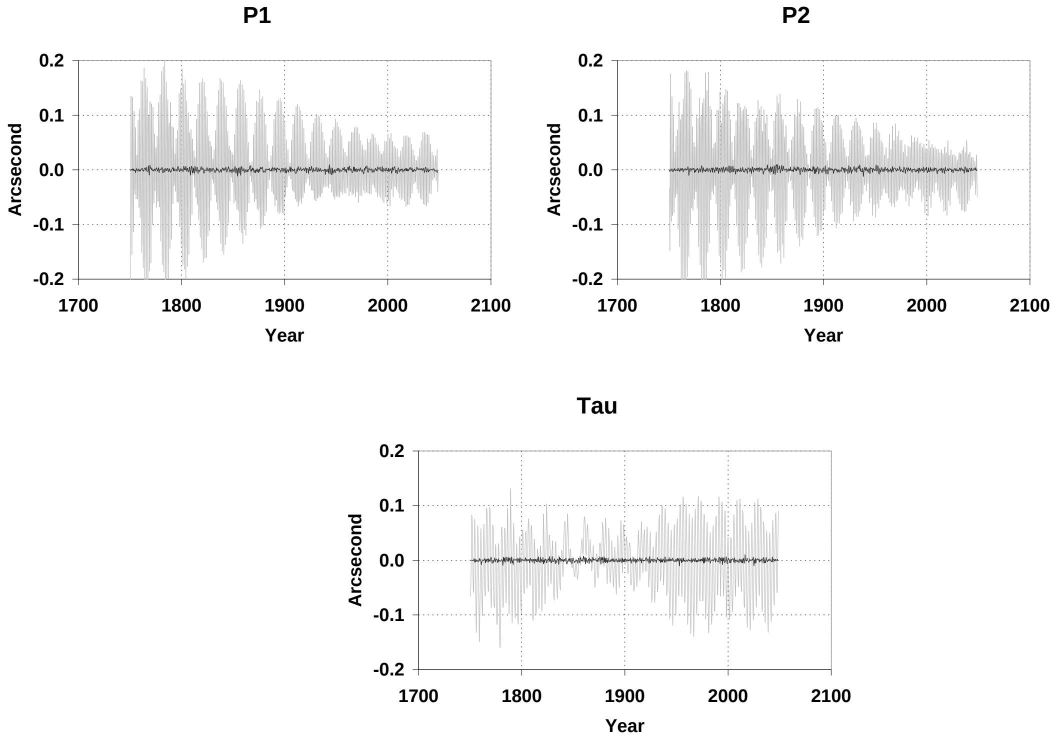

Figure 4: Improvement of the differences $DE_{245} - S_{245}$ with Poisson series; (a) light grey: $DE_{245} - S_{245}$; (b) black: $DE_{245} - (S_{245} + P_{245})$

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