Unified approach of Shapiro and lensing effects in the field of an axisymmetric spinning body

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Introduction

The notion of time transfer function is of crucial interest for modeling light rays in metric theories of gravity.

In a space-time $(\mathcal{V}_4, g)$ with coordinates $x^0 = ct, \boldsymbol{x} = (x^i)$, consider a light ray Γ_{AB} emitted at point $(x^0_A = ct_A, \boldsymbol{x}_A)$ and received at point $(x^0_B = ct_B, \boldsymbol{x}_B)$. We suppose that light propagates through a vacuum, so the ray Γ_{AB} is a null geodesic. The coordinate travel time of light $t_B - t_A$ is a function of $\boldsymbol{x}_A$, $\boldsymbol{x}_B$ and t_B associated to Γ_{AB}:

$$t_B - t_A = \mathcal{T}_\Gamma(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) \qquad \text{we call } \mathcal{T}_\Gamma \text{ a "time transfer function" (TTF)}$$

Knowing TTFs is sufficient to (see, e.g., Hees et al 2014, Bertone et al 2017)

- synchronize distant clocks
- calculate the time delay and Doppler tracking in the Solar System for tests of GR designed to determine the post-Newtonian parameter γ
- determine the gravitational deflection of light in highly precise astrometry (VLBI, Gaia,...)

Introduction

To determine a TTF, it may be assumed that

Assump. 1: $$g_{\mu\nu}(x, G) = \eta_{\mu\nu} + \sum_{n=1}^{\infty} G^n g_{\mu\nu}^{(n)}(x), \quad G = \text{gravitational const.}$$

Assump. 2: $$\underbrace{\mathcal{T}_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B, G) = \frac{|\boldsymbol{x}_B - \boldsymbol{x}_A|}{c} + \sum_{n=1}^{\infty} G^n \mathcal{T}_r^{(n)}(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B)}$$

$$\Downarrow$$

$\mathcal{T}_r^{(n)} =$ integral taken along the straight line joining $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$

(Teyssandier & Le Poncin-Lafitte 2008)

Two problems: Assumpt. 2

→ a single TTF, which is not realist in generic configurations like conjunctions and gravitational lensing;

→ this TTF involves 'enhanced terms'.

Shapiro's formula for the time delay

An illustrating example: TTF for a Schwarzschild-like metric of a spherically symmetric body of mass M within the linearized, weak field approximation

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left\{(dx^0)^2 - \left[1 + \frac{2(\gamma+1)m}{r}\right] d\boldsymbol{x}^2\right\}, \quad m = \frac{GM}{c^2}, \quad r = |\boldsymbol{x}|, \tag{1}$$

where γ is the PN parameter characterizing the curvature of space. The previous assumptions lead to the well-known Shapiro formula (Shapiro 1964):

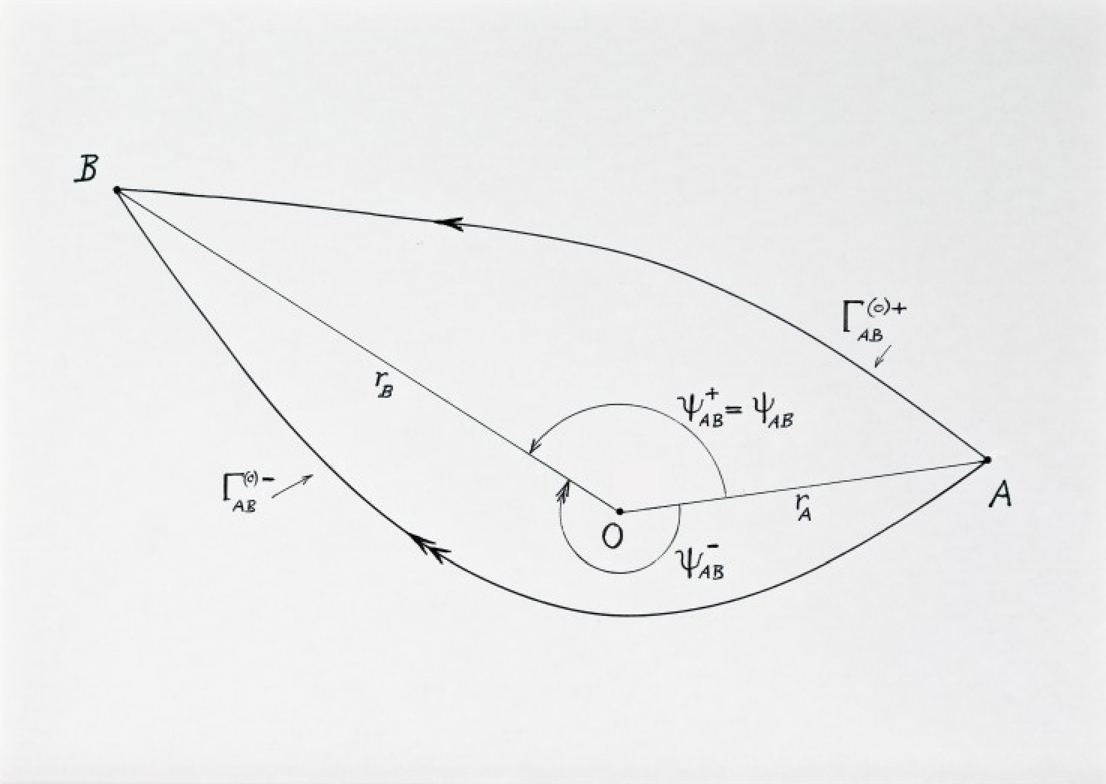
$$c\mathcal{T}_{Sh}(\boldsymbol{x}_A, \boldsymbol{x}_B) = R_{AB} + 2(\gamma+1)m \ln\left(\frac{r_A + r_B + R_{AB}}{2\sqrt{r_A r_B}\cos\frac{\psi_{AB}}{2}}\right) + \mathcal{O}(m^2),$$

where $R_{AB} = |\boldsymbol{x}_B - \boldsymbol{x}_A|$ and ψ_{AB}=angle between $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ defined by the conditions

$$\cos\psi_{AB} = \frac{\boldsymbol{x}_A}{r_A}\cdot\frac{\boldsymbol{x}_B}{r_B}, \quad 0 \leq \psi_{AB} \leq \pi.$$

Enhancement effect: $\mathcal{T}_{Sh}(\boldsymbol{x}_A, \boldsymbol{x}_B) \rightarrow \infty$ when $\psi_{AB} \rightarrow \pi$.

Shapiro's formula for the time delay



It is not surprising: if ψ_{AB} is sufficiently close to π, the straight line joining $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ is passing through a region of strong field.

$\Rightarrow$ a convergent series in powers of G cannot be expected for a TTF in this case!

So, the usual theory is not sufficient and must be seriously improved.

New formulas for Schwarzschild-like metrics

A first step towards a solution: consists in finding the rigorous solutions to the null geodesic eqs of the Schwarzschild-like metric (1) (Linet & Teyssandier 2016).

- There exist two light rays joining $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, denoted by

  $\Gamma_{AB}^{(0)+}$ when the polar angle φ is such that $0 \leq \varphi \leq \psi_{AB} (\leq \pi)$,

  $\Gamma_{AB}^{(0)-}$ when the polar angle is such that $\psi_{AB} - 2\pi \leq \varphi \leq 0$.

- $\Gamma_{AB}^{(0)+}$ and $\Gamma_{AB}^{(0)}$ are Keplerian hyperbolas.
- The full corresponding TTFs can be expressed in a closed form :

$$
c\overline{\mathcal{T}}^{\pm}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{2}\left(\sqrt{r_A + r_B + R_{AB}}\sqrt{r_A + r_B + R_{AB} + 4(\gamma+1)m} \right.
$$

$$
\left. \mp \sqrt{r_A + r_B - R_{AB}}\sqrt{r_A + r_B - R_{AB} + 4(\gamma+1)m}\right)
$$

$$
+2(\gamma+1)m \ln\left(\frac{\sqrt{r_A + r_B + R_{AB} + 4(\gamma+1)m} + \sqrt{r_A + r_B + R_{AB}}}{\sqrt{r_A + r_B - R_{AB} + 4(\gamma+1)m} \pm \sqrt{r_A + r_B - R_{AB}}}\right)
$$

- These TTFs are regular : no enhanced term!

New formulas for Schwarzschild-like metrics

The relevant TTF for the missions in the Solar System is $c\overline{\mathcal{T}}^{+}$, which corresponds to $0 \leq \varphi \leq \psi_{AB} \leq \pi$. It may be expanded in a series of powers of m iff

$$1 + \cos\psi_{AB} \geq \frac{4(\gamma+1)m[r_A + r_B - 2(\gamma+1)m]}{r_A r_B}$$

Then:

$$c\overline{\mathcal{T}}^{+}(\boldsymbol{x}_A, \boldsymbol{x}_B) = R_{AB} + \underbrace{(\gamma+1)m \ln\left[\frac{r_A + r_B + R_{AB} + (\gamma+1)m}{r_A + r_B - R_{AB} + (\gamma+1)m}\right]}_{\text{Shapiro's formula supplemented with Moyer's terms}}$$

$$\underbrace{+\frac{(\gamma+1)^3 m^3 R_{AB}(r_A + r_B)}{2r_A^2 r_B^2 (1+\cos\psi_{AB})^2} - \frac{(\gamma+1)^4 m^4 [3(r_A+r_B)^2 + R_{AB}^2]}{3r_A^3 r_B^3 (1+\cos\psi_{AB})^3} + \mathcal{O}(m^5)}_{\text{These divergent terms when } \psi_{AB}\to\pi \text{ have no physical meaning}}$$

We recover the 'enhanced terms' of 3rd and 4th orders taken into account for extracting the well-known γ estimate from Cassini mission (Ashby & Bertotti 2009 2010). For a justification of Moyer's term, see Klioner & Zschocke 2010.

A numerical example

For $r_A = 5\,\text{au}$, $r_B = 1\,\text{au}$ and $r_c = r_A r_B \sin\psi_{AB}/R_{AB} = 2R_\odot$, $R_\odot$ being the radius of the Sun :

$$\Delta\gamma = 10^{-8} \quad \Longrightarrow \quad \Delta\overline{\mathcal{T}}^{+} = 1.22 \times 10^{-12}\,\text{s}$$

to compare with

$$\left(\overline{\mathcal{T}}^{+} - \mathcal{T}_{\text{Moyer}}\right)_{\gamma=1} = 0.71 \times 10^{-12}\,\text{s} \tag{3}$$

Note that

$$\left(\overline{\mathcal{T}}^{+} - \mathcal{T}_{\text{Shap}}\right)_{\gamma=1} = -3.74 \times 10^{-9}\,\text{s}$$

is 3×10^3 greater than the effect to detect!

The full closed form of $\overline{\mathcal{T}}^{+}$ will be relevant in the missions aiming to measure γ with an accuracy of 10^{-8}.

Generalization: TTFs for a spinning axisymmetric body

The next step: extension to a non spherical axisymmetric body spinning around its axis of symmetry, with a metric of the form

$$ds^2 = \left(1 - \frac{2W}{c^2}\right)\left\{(dx^0)^2 + \frac{4(\gamma+1)}{c^3}(\boldsymbol{W}.\boldsymbol{dx})dx^0 - \left[1 + \frac{2(\gamma+1)W}{c^2}\right]\right\} d\boldsymbol{x}^2, \tag{4}$$

where

$$\boxed{\frac{1}{c^2}W(\boldsymbol{x}) = \frac{m}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{r_0}{r}\right)^n P_n(\boldsymbol{s}.\boldsymbol{n})\right]},$$

$$\boxed{\frac{1}{c^3}\boldsymbol{W}(\boldsymbol{x}) = \frac{ma(\boldsymbol{s}\times\boldsymbol{x})}{2r^3}\left[1 - \sum_{n=1}^{\infty} K_n \left(\frac{r_0}{r}\right)^n P'_{n+1}(\boldsymbol{s}.\boldsymbol{n})\right]},$$

with

$$\boxed{\boldsymbol{n} = \frac{\boldsymbol{x}}{r}, \quad \boldsymbol{s} = \frac{\boldsymbol{S}}{|\boldsymbol{S}|}, \quad a = \frac{S}{Mc} = \frac{GS}{mc^3} \text{ (Kerr parameter).}}$$

and P_n being the Legendre polynomial of degree n. $\boldsymbol{S}$ is the angular momentum, the J_n and K_n are the mass-multipole and spin-multipole moments, respectively.

TTFs in the field of a spinning axisymmetric body

Fundamental assumption: We assume that to $\Gamma_{AB}^{(0)+}$ (resp. $\Gamma_{AB}^{(0)-}$), one may associate a time transfer function $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) such that

$$\mathcal{T}^{\pm}(\boldsymbol{x}_A, \boldsymbol{x}_B; J_n, \boldsymbol{S}, K_n) = \overline{\mathcal{T}}^{\pm}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \sum_n J_n \Delta\mathcal{T}_{J_n}^{\pm}(\boldsymbol{x}_A, \boldsymbol{x}_B) + a\Delta\mathcal{T}_{\boldsymbol{S}}^{\pm}(\boldsymbol{x}_A, \boldsymbol{x}_B)$$
$$+ \sum_n K_n \Delta\mathcal{T}_{K_n}^{\pm}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \cdots ,$$

where the $\Delta\mathcal{T}^{\pm}$ are perturbation terms of the first order in J_n, $\boldsymbol{S}$ and K_n.

- Since $\mathcal{T}^{\pm}$ satisfies the eikonal equation (Teyssandier & Le Poncin 2008):

$$g^{00}(\boldsymbol{x}) - 2g^{0i}(\boldsymbol{x})\frac{\partial\mathcal{T}^{\pm}}{\partial x^i} + g^{ij}(\boldsymbol{x})\frac{\partial\mathcal{T}^{\pm}}{\partial x^i}\frac{\partial\mathcal{T}^{\pm}}{\partial x^j} = 0, \tag{5}$$

  each $\Delta\mathcal{T}^+$ (resp. $\Delta\mathcal{T}^-$) can be expressed by a simple integral taken along the unperturbed light ray $\Gamma_{AB}^{(0)+}$ (resp. $\Gamma_{AB}^{(0)-}$).

- These integrals can be calculated with any symbolic computer program.

Contribution of J_2

For light rays in the equatorial plane, i.e. when $\boldsymbol{s}.\boldsymbol{n}_A = \boldsymbol{s}.\boldsymbol{n}_B = 0$:

$$c\Delta\mathcal{T}_{J_2}^{\pm}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{2}(\gamma+1)mJ_2\left(\frac{r_0}{b_\pm}\right)^2\left[b_\pm\frac{1-\cos\psi_{AB}}{\sin\psi_{AB}}\left(\frac{1}{r_A}+\frac{1}{r_B}\right)\right.$$
$$\left.+\frac{(\gamma+1)m}{b_\pm}\left(\psi_{AB}^{\pm}-2\frac{1-\cos\psi_{AB}}{\sin\psi_{AB}}\right)\right],$$

$b^{\pm}$ = impact parameter of $\Gamma_{AB}^{(0)\pm}$ (cf. Appendix), $\quad \psi_{AB}^{+} = \psi_{AB}, \quad \psi_{AB}^{-} = \psi_{AB} - 2\pi.$

No enhanced term, since

$$\lim_{\psi_{AB}\to\pi}\left[c\Delta\mathcal{T}_{J_2}^{\pm}(\boldsymbol{x}_A, \boldsymbol{x}_B)\right] = \frac{1}{2}J_2\frac{r_0^2(r_A+r_B)}{r_Ar_B}\left[\sqrt{1+\frac{2(\gamma+1)m}{r_A+r_B}}+\frac{\pi}{2}\sqrt{\frac{(\gamma+1)m(r_A+r_B)}{2r_Ar_B}}\right]$$

Conclusion contrasting with the expression

$$c\Delta\mathcal{T}_{J_2}^{+}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{2}(\gamma+1)mJ_2\frac{r_0^2}{r_Ar_B}\frac{(r_A+r_B)R_{AB}}{r_Ar_B(1+\cos\psi_{AB})}$$

found in the litterature (Klioner & Kopeikin 1992, Linet & Teyssandier 2002, Le Poncin-Lafitte & Teyssandier 2008, Soffel & Han 2015,...).

Conclusion

In a previous paper, a complete description of the light rays deflected by a static spherically symmetric body was developped within the linearized, weak field approximation. As major results:

- Shapiro's formula for the time delay is replaced by an analytical expression devoid of divergence;

- The difference between the Moyer formula and the new one will have to be taken into account for modeling the determination of γ at a level of accuracy of 10^{-8} and the nano-arcsecond astrometry.

An extension of this analysis to light rays propagating in the field of a spinning axisymmetric body is in preparation.

- The first-order perturbation terms in TTFs due to the mass and the spin multipoles are given by simple integrals taken along the unperturbed light rays.

- These terms can be explicitly calculated with any symbolic computer program.

- They are devoid of any divergence or enhancement effect.

Appendix: the impact parameters $b_\pm$

Intrinsic impact parameters of the light rays $\Gamma_{AB}^{(0)+}$ and $\Gamma_{AB}^{(0)+}$ (Linet & Teyssandier 2016) :

$$b_\pm = \frac{r_A r_B}{\sqrt{2} R_{AB}} \sin\frac{\psi_{AB}}{2} \left[\sqrt{1+\cos\psi_{AB} + \frac{2(\gamma+1)m(r_A+r_B-R_{AB})}{r_A r_B}} \pm \sqrt{1+\cos\psi_{AB} + \frac{2(\gamma+1)m(r_A+r_B+R_{AB})}{r_A r_B}} \right]$$

When ψ_{AB} is small, only b_+ is relevant:

$$b_+ = r_c \left[1 + \frac{(\gamma+1)m(r_A+r_B)}{r_A r_B(1+\cos\psi_{AB})} + \mathcal{O}(m^2) \right]$$

Limit of $b_\pm$ when $\psi_{AB} \to \pi$:

$$\lim_{\psi_{AB}\to\pi} b_\pm = \pm\sqrt{\frac{2(\gamma+1)\, m\, r_A r_B}{r_A + r_B}}$$

References

- Ashby N & Bertotti B 2009 *Class. Quantum Grav.* **26** 094001.
- Ashby N & Bertotti B 2010 *Class. Quantum Grav.* **27** 145013.
- Bertone S *et al* 2017 *A & A* **608** A 83.
- Hees A, Bertone S & Le Poncin-Lafitte C 2014 *Phys. Rev. D* **89** 064045.
- Klioner S A & Kopeikin SM 1992 *Astron. J.* **104** 887.
- Klioner S A 2003 *Astron. J.* **125** 1580.
- Klioner S A & Zschocke S 2010 *Class. Quantum Grav.* **27** 075015.
- Le Poncin-Lafitte C, Linet B & Teyssandier P 2004 *Class. Quantum Grav.* **21** 4463.
- Le Poncin-Lafitte C & Teyssandier P 2008 *Phys. Rev. D* **77** 044029.
- Linet B & Teyssandier P 2002 *Phys. Rev. D* **66** 024045.
- Linet B & Teyssandier P 2016 *Phys. Rev. D* **93** 044028.
- Shapiro I I 1964 *Phys. Rev. Lett.* **13** 789.
- Soffel M H & Han W-B 2015 *Physics Letters A* **379** 233.
- Teyssandier P& C. Le Poncin-Lafitte 2008 *Class. Quantum Grav.* **25** 145020.