On the Extension of Adams–Bashforth–Moulton Methods for Numerical Integration of Delay Differential Equations and Application to the Moon's Orbit

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Types of differential equations

Ordinary differential equation (ODE):

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t))$$

(Retarded) delay differential equation (DDE):

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(\varphi(t)), ...),\ \varphi(t) < t$$

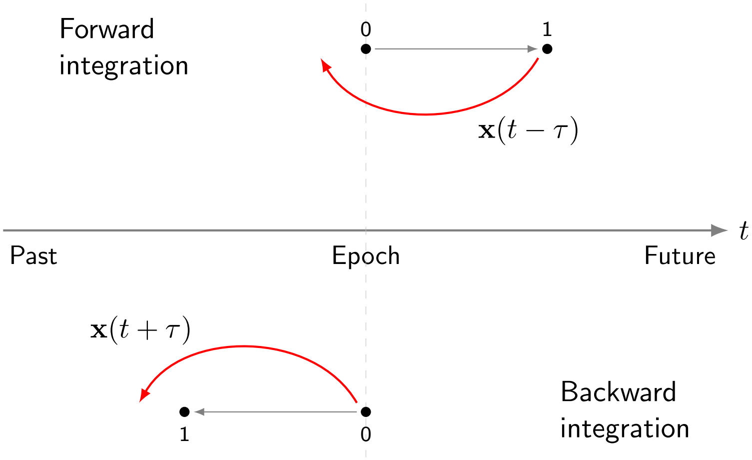
Advanced differential equation:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(\psi(t)), ...),\ \psi(t) > t$$

DDE of neutral type:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \dot{\mathbf{x}}(\xi(t)), ...),\ \xi(t) \neq t$$

From retarded to advanced equations



Rotational equation of the Moon (general form)

Forward: retarded DDE of neutral type with constant delays

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t-\tau), \dot{\mathbf{x}}(t-\tau))$$

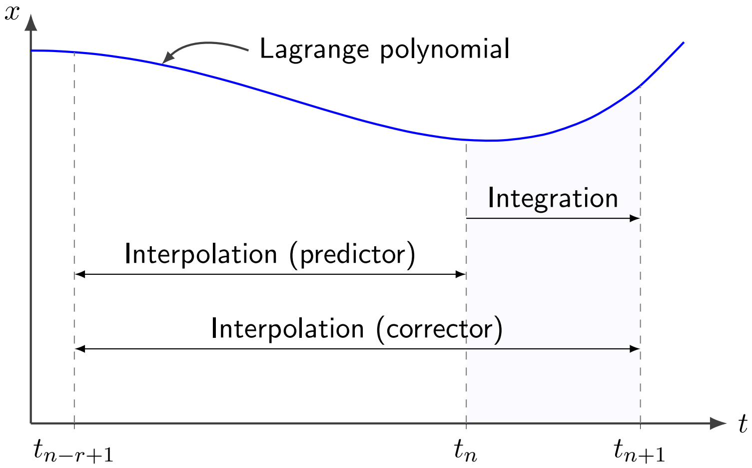
Backward: advanced DDE of neutral type with constant delays

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t+\tau), \dot{\mathbf{x}}(t+\tau))$$

Initial condition at the epoch:

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

Adams–Bashforth–Moulton methods (I)



Adams–Bashforth–Moulton methods (II)

1. Predictor — Adams–Bashforth (order 2):

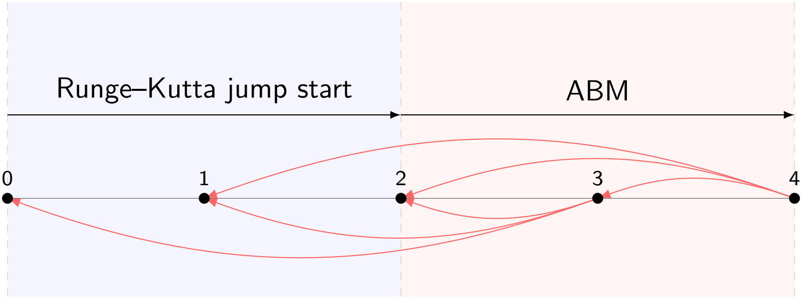
$$\mathbf{x}_{n+2} = \mathbf{x}_{n+1} + h\left(\frac{3}{2}\mathbf{f}_{n+1} - \frac{1}{2}\mathbf{f}_n\right)$$

2. Evaluation of $\mathbf{f}_{n+2} = \mathbf{f}(t_{n+2}, \mathbf{x}_{n+2})$
3. Corrector — Adams–Moulton (order 3):

$$\mathbf{x}_{n+2} = \mathbf{x}_{n+1} + h\left(\frac{5}{12}\mathbf{f}_{n+2} + \frac{2}{3}\mathbf{f}_{n+1} - \frac{1}{12}\mathbf{f}_n\right)$$

4. (Optional). PECE, PECEC, PECECE

Adams–Bashforth–Moulton methods (III)



First $(r-1)$ steps must be performed by a single-step method.

The ‘nested RK4’ method for DDEs

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t \pm \tau), \dot{\mathbf{x}}(t \pm \tau))$$
$$\mathbf{x}(t_0) = \mathbf{x}_0$$

1. Introduce a new function

$$\mathbf{g}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t), \dot{\mathbf{x}}(t))$$

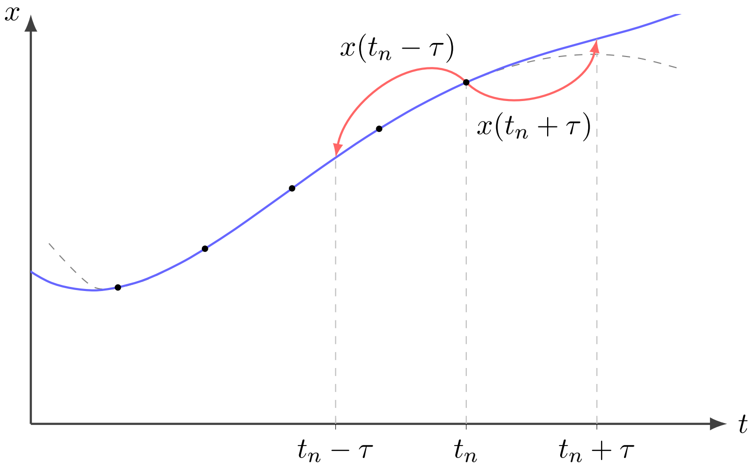
2. Retrieve delayed states by integrating $\mathbf{g}(t)$ with RK4

$$\mathbf{x}(t \pm \tau) = {}^{\mathrm{RK4}}\mathcal{I}_{t \to t \pm \tau} \mathbf{g}(t)$$
$$\dot{\mathbf{x}}(t \pm \tau) = \mathbf{g}(t \pm \tau)$$

Drawbacks

- Calculation of each delayed state requires 4 RHS calls
- No previous knowledge of $\mathbf{x}(t)$ is being used

The interpolation method for DDEs (I)



The interpolation method for DDEs (II)

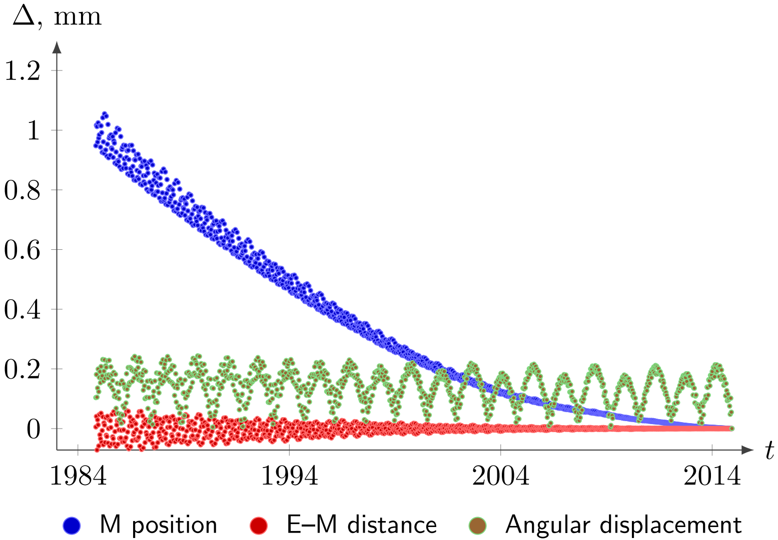
The algorithm

1. Jump start by the 'nested RK4' algorithm (with 8th order Dormand–Prince)
2. P and C stages are simple ABM
3. Each E stage constructs a Lagrange interpolating polynomial (any order), which is used to find $\mathbf{x}(t \pm \tau)$ and $\dot{\mathbf{x}}(t \pm \tau)$

Advantages

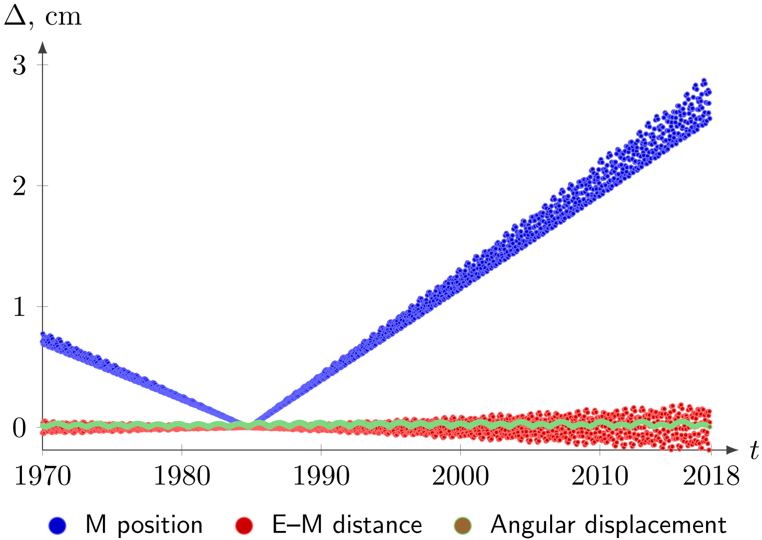
- Much cheaper delayed states
- No simplifying assumptions about the function

Results. The forward–backward test



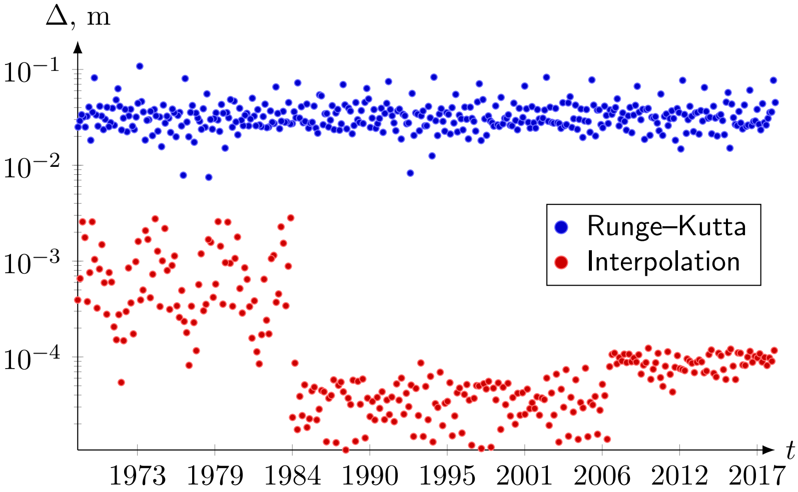
Orders: ABM — 13, interpolation and extrapolation — 9

Results. The old–new test



Orders: ABM — 13, interpolation and extrapolation — 9

Results. Approximation accuracy

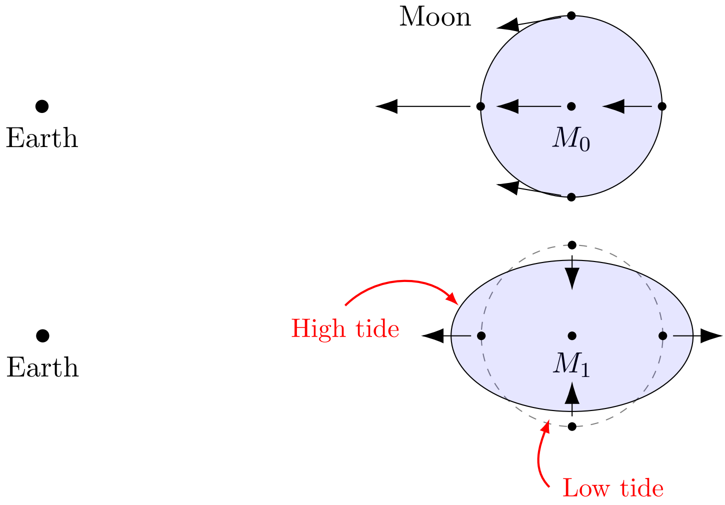


Results. O−C

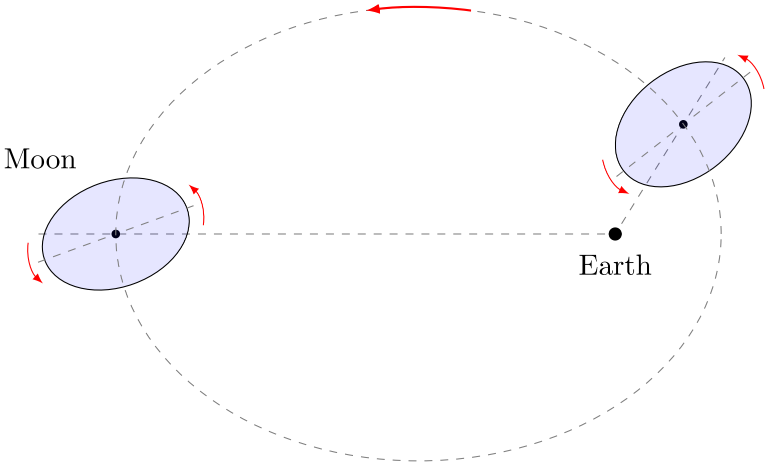
Station	Timespan	NPs	One-way WRMS, cm
McDonald	1969–1985	3552/52	19.9
MLRS1	1983–1988	588/43	11.2
MLRS2	1988–2015	3216/454	3.4
Haleakala	1984–1990	748/22	5.7
OCA (Ruby)	1984–1986	1109/79	17.0
OCA (YAG)	1987–2005	8203/121	2.0
OCA (MeO)	2009–2017	1814/22	1.42
OCA (IR)	2015–2017	1814/20	Old: 1.27 New: 1.26
APOLLO	2006–2016	2609/39	1.37

Backup slides

Tidal forces (I)



Tidal forces (II)



Rotational equation of the Moon (actual form)

Euler's equation for a rotating reference frame:

$$\dot{\boldsymbol{\omega}} = \left(\frac{I}{m}\right)^{-1} \left[\frac{\mathbf{N}}{m} - \frac{\dot{I}}{m}\boldsymbol{\omega} - \boldsymbol{\omega} \times \left(\frac{I}{m}\boldsymbol{\omega}\right)\right],$$

$\boldsymbol{\omega}$ — angular velocity,
$\mathbf{N}(t)$ — torque,
I/m — inertia tensor

$$\frac{I}{m} = \frac{I_0}{m} - \frac{I_c}{m} - k_2 \frac{\mu_E}{\mu_M}\left(\frac{R_M}{r}\right)^5 \begin{bmatrix} x^2 - \frac{1}{3}r^2 & xy & xz \\ xy & y^2 - \frac{1}{3}r^2 & yz \\ xz & yz & z^2 - \frac{1}{3}r^2 \end{bmatrix}$$
$$+ k_2 \frac{R_M^5}{3\mu_M} \begin{bmatrix} \omega_x^2 - \frac{1}{3}(\omega^2 - n^2) & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & \omega_y^2 - \frac{1}{3}(\omega^2 - n^2) & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & \omega_z^2 - \frac{1}{3}(\omega^2 + 2n^2) \end{bmatrix},$$

where $\mathbf{r} = (x, y, z)^{\mathrm{T}} = \mathbf{r}(t - \tau),\ \boldsymbol{\omega} = \boldsymbol{\omega}(t - \tau),\ \tau = 0.096\ \mathrm{d}$

Runge–Kutta methods

General form for the Runge–Kutta family of methods:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h \sum_{i=1}^{s} b_i \mathbf{k}_i$$

$$\mathbf{k}_s = \mathbf{f}(t_n + c_s h, \mathbf{x}_n + h \sum_{j=1}^{s-1} a_{s,j} \mathbf{k}_j)$$

Drawbacks

- Butcher barriers:
  - $p \geq 5$: no RK method exists of order p with $s = p$ stages
  - $p \geq 7$: no RK method exists of order p with $s = p + 1$ stages
  - $p \geq 8$: no RK method exists of order p with $s = p + 2$ stages
- Higher orders are problematic

Other methods

1. **Pavlov, IAA RAS**

   Nested Runge–Kutta method (described above)

2. **Hofmann, Institut für Erdmessung**

   Quadratic Taylor expansion using $\mathbf{x}$, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$

3. **Williams, NASA JPL**

   Approximation with pre-fit polynomials (exact algorithm unknown)