SECOND ORDER EFFECTS IN IAU2000 NUTATION MODEL

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ABSTRACT. Second order effects, in the sense of perturbation theories, of IAU2000 nutation model (Mathews et al. 2002) are inherited from the Hamiltonian rigid Earth nutations REN2000 (Souchay et al. 1999). The transformation to IAU2000 non-rigid Earth model is made by applying the same frequency-dependent transfer function as in the case of first order nutations.

We analyze the nature of the second order effects considered in REN2000 and the used way to derive their corresponding non-rigid contributions. In addition, we discuss the existence of some additional second order terms that, in contrast to the rigid model, might play a role for the non-rigid Earth. The situation is exemplified for a Poincaré non-rigid Earth model, obtaining the second order nutations of the angular momentum axis (Poisson terms) by means of a Hamiltonian approach.

1. INTRODUCTION

Current accuracy demands in the Earth rotation modeling require the incorporation of terms previously neglected (e.g., Ferrándiz et al. 2020). Among them, one of the most important group is that referred to as second order terms. In fact, some of those terms were considered in the first works of the modern Earth rotation theories (e.g., Kinoshita 1977 or Kinoshita & Souchay 1990).

The nature of second order effects, however, is not uniform. We can distinguish between physical and mathematical second order terms. The first ones are due to interactions, Earth model features, etc. that have a small magnitude with respect to a reference value associated to the Earth rotational dynamics —typically the kinetic energy in the free rotational motion. Some representatives are higher order terms of the geopotential, direct effects of the inner core, etc. Once modeled, they can be incorporated into the theory following a standard first order, or linear, procedure.

The second ones are related to our (un)skill to solve the differential equations of the rotational motion. They emerge as a consequence of developing a more approximate solution to those equations. Their determination is cumbersome, specially if one is interested in obtaining analytical solutions, and many techniques have been historically developed to tackle with those terms (e.g., Ferraz–Mello 2007). Within our context the names second order effects in the sense of perturbation theories, nutation-nutation coupling, crossed-nutation effect, etc. belong to this kind of contributions and are equivalent.

In this article, we aim at sketching how these mathematical second order terms are currently incorporated in IAU2000 nutation model and why their treatment is neither consistent nor complete.

2. SECOND ORDER TERMS IN IAU2000

IAU2000 nutation model (Mathews et al. 2002) is based on a transfer function —or normalized amplitude— characterizing the features of the Earth model under consideration (three layers, anelastic mantle, etc.) which is applied to the rigid Earth Hamiltonian nutation series REN2000 (Souchay et al. 1999).

By doing so, the non-rigid nutations due to the lunisolar torque are obtained from the rigid nutations of the Earth figure axis[1] described by the rigid amplitude nutation $\tilde{\eta}_R(t)$. This is accomplished by the product of $\tilde{\eta}_R(\sigma)$ with the transfer function $T(\sigma)$ in the frequency domain, where σ denotes the frequency of any spectral component of the gravitational potential relative to the rotating Earth and stemming from the orbital motion of the Moon and the Sun —see Mathews et al. (1991) for further details.

This way of obtaining the nutations is very useful, since it uses the rigid model as a proxy, avoiding the direct manipulation of the geopotential. In its derivation, however, it is implicitly assumed a first order theory of the rigid Earth (e.g., Mathews et al. 1991) what represents a limitation for its general application to higher orders of perturbation.

Indeed, the procedure of construction of the transfer function considers a relationship between the polar motion of the rigid Earth $\tilde{m}_R(\sigma)$ and the tesseral part of the second degree of the gravitational potential of the Earth $\tilde{\phi}(\sigma)$ given by

$$\tilde{m}_R(\sigma) = \frac{e}{e-\sigma}\tilde{\phi}(\sigma). \tag{1}$$

Here, $e = (C-A)/A$ is the ellipticity of the Earth with A the principal moment of inertia about any axis contained in the equatorial plane —passing through the Earth's center of mass— and C about the axis perpendicular to it.

Equation (**??**) is similar to the first order theory developed in Eqs. (6.22) and (6.23) by Kinoshita (1977), as it can be shown by a proper identification of the notations. In this regard, the Hamiltonian framework employs customarily the dynamical ellipticity $H_d = (C-A)/C$ instead of e. That dynamical ellipticity is included in the parameter k —k_M and k_S— related to the perturbers and considered in those theories. Specifically, we have

$$k_{M,S} = 3\frac{GM_{M,S}}{a^3_{M,S}\omega_E}H_d. \tag{2}$$

The linear e dependence in the numerator of Eq. (**??**) —alternatively, the H_d dependence— is due to the linear response of the polar motion to the geopotential, valid for a first order theory. The functional form of the denominator is associated with the proper modes of the Earth model. Hence, in the case of a rigid Earth it just involves the Eulerian frequency.

When moving to non-rigid Earth models the number of proper modes increases and Eq. (**??**) is substituted by

$$\tilde{m}(\sigma) = \left[M^{-1}(\sigma)\,y(\sigma)\right]_1\tilde{\phi}(\sigma). \tag{3}$$

In this way, the transfer function is given by (Mathews et al. 1991)

$$T(\sigma) = \frac{\tilde{\eta}(\sigma)}{\tilde{\eta}_R(\sigma)} = \frac{\tilde{m}(\sigma)}{\tilde{m}_R(\sigma)} = \frac{e-\sigma}{e}\left[M^{-1}(\sigma)\,y(\sigma)\right]_1, \tag{4}$$

which keeps the linear dependence with e in the denominator as derived from Eq. (**??**). It allows the computation of the non-rigid amplitude by

$$\tilde{\eta}(\sigma) = T(\sigma)\,\tilde{\eta}_R(\sigma). \tag{5}$$

The current standard of the Earth nutation IAU2000 (Mathews et al. 2002) applies the former procedure (Eqs. **??** & **??**) to the total rigid nutation amplitudes of the figure axis due to the lunisolar perturbation (Souchay et al. 1999). However, those amplitudes result from different effects.

In particular, one part of the rigid terms is due to second order effects in the sense considered in this work (Souchay et al. 1999, Table 1). They take into account two main contributions:

[1]The situation is different for the nutations of planetary origin as it has been recently shown in Ferrándiz et al. 2018).

- Crossed-nutations: characterized by the influence of the nutation itself on the torque exerted by the Moon and the Sun. This is the most important part and it is intrinsically associated to the rotation —rotation on rotation effects.

- Spin-orbit coupling: it is due to the interaction between the orbital motion of the Moon and the J_2 component of the geopotential. This effect is mainly related to the way in which the Moon ephemeris (ELP-2000, Chapront-Touzé & Chapront 1983) are used when constructing the rotation theory of the Earth.

Since the second order nutations are expected to be small, REN2000 (Souchay et al. 1999) performs different simplifications that make easier the computations. One of the most important is the identification of the amplitudes of the figure axis with those of the angular momentum axis (Poisson terms), i.e., it neglects contributions related to the Oppolzer terms. Those approximations are right from a numerical point of view —at the 2 μas level— as it was shown in the comprehensive second order theory constructed in Getino at al. (2010), where the main part of those simplifications are removed.

As a consequence, the amplitudes in longitude and obliquity of the angular momentum axis provided in REN2000 (Souchay et al. 1999; sections 2 and 3) depend mainly on the orbital characteristics of the perturbers, which are known functions of time provided by the corresponding orbital ephemeris, but not on the Earth model.

The only way in which the Earth structure enters into these expressions is through a linear dependence with the parameter H_d^2, not H_d. There is no dependence of those formulae on the proper mode of the rigid model which would be introduced through the Eulerian frequency as it is the case in the first order expressions (e.g., Kinoshita 1977, Eqs. 6.22 and 6.23 through N_g).

The application of the transfer function approach, as done in IAU2000, under these circumstances gives raise, at least, to the main following problems:

- The transfer function given in Eq. (**??**) cannot be applied to second order terms, since the second order contributions to the polar motion are proportional to e^2 (Getino et al. 2010, Eqs. 69) and not to e like in Eq. (**??**)

- Even if it were correct, it cannot be applied to REN 2000 (Souchay et al. 1999) second order terms, because they do not depend on Earth structure (not consistent). They just require a scaling of the form H_d^2/H_{Rd}^2 to take into account the change in the dynamical ellipticity value when passing from the rigid to the non-rigid Earth model

Those facts, although numerically small, represent inconsistencies in IAU2000 that must be avoided.

In addition, IAU2000 totally lacks from the effect of Earth's structure on the second order terms, simply because it is the case of REN2000 (Souchay et al. 1999). Hence, all the second order contributions to the Oppolzer terms are absent. It is also the case of the part of the second order amplitudes of Poisson that depends on the Earth model.

Due to the fluid core resonance those terms can be amplified, contributing in a non-negligible way in view of current accuracies as it has been shown in the case of precession (Baenas et al. 2017). This situation is summarized in Table **??**.

3. POINCARÉ EARTH MODEL: POISSON TERMS

To solve the former difficulties concerning the construction of a second order theory of the non-rigid Earth two main steps are required. First, it is necessary to develop a framework where the second order terms can be derived in a consistent way, since the current transfer function procedure is not valid at the second order. Second, we have to compute the second order amplitudes for different Earth models, evaluating the real contribution of the non rigidity to the nutations through the normal modes of the considered non-rigid Earth.

Second order terms	REN2000	IAU2000
Poisson Model independent (but H_d^2)	Present	Incorrect modeled (**not consistent**)
Poisson Model dependent	Absent	Absent (**incomplete**)
Oppolzer Model dependent	Absent	Absent (**incomplete**)

Table 1: Second order terms considered in IAU2000.

Both can be accomplished following a Hamiltonian approach, since this formalism is naturally fitted to construct analytical approximate solutions of the second order by means of perturbation theories. Indeed, the same approach was used for the rigid Earth in REN2000 (Souchay et al. 1999) and later extended by Getino et al. (2010).

The procedure, even with the use of symbolic software, is quite cumbersome due to the intrinsic complexity of second order theories and to the number of degrees of freedom of the non-rigid models. Hence, we have started this study considering the second order nutations for the Poisson terms of a Poincaré Earth model —rigid mantle and fluid core. The developments are out of the scope of this contribution and are presented in detail in Getino et al. (2020).

They are based on considering an specific Non Singular Complex Canonical Variables (NSCCV) set combined with a perturbation theory based on canonical transformations (Hori 1966). The use of the NSCCV set allows obtaining an Hori kernel that simplifies the application of the perturbation algorithm up to the second order.

That procedure leads to the determination of second order analytical expressions for the nutations of the angular momentum axis. The most important conclusion is that, in contrast to first order results, Poisson terms do depend on the Earth interior structure. In our case, that dependence arises from the normal modes of the Poincaré model, i.e., the Chandler Wobble (CW) and the Free Core Nutation (FCN).

The general structure of those nutations are

$$\begin{aligned}
\Delta_2\lambda = \quad & H_d^2 \sum_{p,q} c_p c_q \left[\sum_{\substack{ip\neq 0,\, jq\neq 0 \\ \tau,\,\rho=\pm 1}} \mathcal{L}^a_{i_p, j_q, \tau, \rho} + \sum_{\substack{ip,\, jq \\ \tau,\,\rho=\pm 1}} \mathcal{L}^b_{i_p, j_q, \tau, \rho} \right] \sin\left(\tau\Theta_{i_p} - \rho\Theta_{j_q}\right) , \\
\Delta_2 I = \quad & H_d^2 \sum_{p,q} c_p c_q \left[\sum_{\substack{ip\neq 0,\, jq\neq 0 \\ \tau,\,\rho=\pm 1}} \mathcal{O}^a_{i_p, j_q, \tau, \rho} + \sum_{\substack{ip,\, jq \\ \tau,\,\rho=\pm 1}} \mathcal{O}^b_{i_p, j_q, \tau, \rho} \right] \cos\left(\tau\Theta_{i_p} - \rho\Theta_{j_q}\right) ,
\end{aligned} \tag{6}$$

where the amplitudes with superscript a are independent of the Earth model —but H_d^2, and those with superscript b do depend on it. For example, one of the contributions of $\mathcal{L}^a_{i_p, j_q, \tau, \rho}$ and $\mathcal{O}^a_{i_p, j_q, \tau, \rho}$ —model independent— given by

$$\frac{1}{8} \frac{\tau m_{5i}}{\tau n_i - \rho n_j} \left(\frac{1}{\tau n_i} + \frac{1}{\rho n_j} \right) \left(\tau m_{5i} B_i B'_j + \rho m_{5j} B'_i B_j \right) . \tag{7}$$

And for $\mathcal{L}^b_{i_p,j_q,\tau,\rho}$ and $\mathcal{O}^b_{i_p,j_q,\tau,\rho}$ —model dependent— by

$$\frac{\sin I}{2}\frac{1}{(\tau n_i-\rho n_j)}\frac{\omega_E-\tau n_i-r_3}{\prod\limits_{k=1,2}(\omega_E-\tau n_i-\sigma_k)}\left(C'_{i,\tau}C_{j,\rho}+C_{i,\tau}C'_{j,\rho}\right). \quad (8)$$

In the former expressions, Θ_k denotes a combination of the Delaunay variables for the Moon and the Sun and n_k represent its time derivative. The orbital functions B and C were introduced in Kinoshita (1977) — see Getino et al. (2020) for a full explanation of the notations.

Equation (**??**) provides basically the same contributions as those given in REN2000 (Souchay et al. 1999), since they are model independent. However, the terms of the form of Eq. (**??**) depend on the Earth model through r_3 and the parameters $\sigma_{1,2}$ that are related to CW and FCN.

For a particular parameter set of a Poincaré model derived from Getino & Ferrándiz (2001), we have evaluated the former formulas, recovering as a limiting case the rigid values. We have also reproduced (not displayed) the second order contribution to the precession rate (Baenas et al. 2017). As it can be seen in Table **??**, the numerical differences, i.e., the second order contributions of the non-rigidity, are relevant for some frequencies at the tens μas level even for the Poisson terms.

Argument					Period	Poincaré Earth		Rigid Earth		Difference	
l_M	l_S	F	D	Ω	(days)	Lon.	Obl.	Lon.	Obl.	Lon.	Obl.
0	0	0	0	1	-6798.36	-27.2	72.0	-30.1	30.0	2.9	42.0
0	0	0	0	2	-3399.18	-1209.0	234.5	-1212.6	236.4	3.6	-1.9
0	1	0	0	0	365.26	0.4	-0.9	1.1	-0.1	-0.7	-0.8
0	0	2	-2	2	182.62	-7.4	3.7	-0.3	0.1	-7.2	3.9
0	0	2	-2	1	177.84	91.9	-72.5	92.6	-73.0	-0.8	0.6
0	0	2	0	2	13.66	-5.7	1.4	-4.9	1.0	-0.9	0.6

Table 2: Second order Poisson terms: In-phase, Poincaré model (Units: μas).

4. CONCLUSIONS

Second order terms in the sense of perturbation theories are not consistently considered by current IAU2000 (Mathews et al. 2002) nutation model. That incorrect modeling might lead to some differences of a few μas —to be determined. It can be corrected by transforming second order rigid amplitudes of REN2000 (Souchay et al. 1999) through a re-scaling of H_d^2. In addition, IAU2000 (Mathews et al. 2002) lacks the influence of the Earth structure (normal modes) in Poisson terms and Oppolzer terms, simply because it was not considered in REN2000 (Souchay et al. 1999).

Even having the complete amplitudes of the second order rigid part (Getino et al. 2010), obtaining the non-rigid contributions with the current approach is not direct, because the used transfer function assumes linearity what is not valid for second order terms.

The Hamiltonian approach provides a suitable framework to derived the second order nutations of a non-rigid Earth. For Poincaré model, we have shown that at the second order Poisson terms are affected by the Earth structure with non-negligible amplitudes (Getino et al. 2020). This approach must be extended to compute Oppolzer terms and incorporated in the standard models of the rotation of the Earth.

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