COMPOSITION OF LORENTZ TRANSFORMATIONS IN TERMS OF THEIR GENERATORS

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Abstract

Two-forms in Minkowski space-time may be considered as generators of Lorentz transformations. Here, the covariant and general expression for the composition law (Baker–Campbell–Hausdorff formula) of two Lorentz transformations in terms of their generators is obtained. For simplicity, the expression is first obtained for complex generators, then translated to real ones. Every generator has two essential eigenvalues and two invariant (two–)planes; the eigenvalues and the invariant planes of the Baker–Campbell–Hausdorff composition of two generators are also obtained.

Key words: Lorentz transformations, Lie algebras of two–forms, Baker–Campbell–Hausdorff formula.

I Introduction

In Minkowski space-time, global Lorentz transformations are used to relate *inertial observers*. In a general space-time, local Lorentz transformations [1], along time-like congruences of curves or space-like families of hypersurfaces, are used to relate arbitrary frames to *comoving observers* [2] or to *synchronizations* [3].

In most of the problems, the transformations involved belong to the *proper orthocronous* Lorentz group (its connected component of the identity), so that they are univocally given by the exponential of the elements of the Lorentz algebra [4]. Local Lorentz transformations of the space-time may thus be given by exponentials of the two-forms [5] of the space-time.

This representation by two-forms of local Lorentz transformations has the important advantage of involving exclusively its *intrinsic elements* [6]; these are the elements in which the corresponding two-forms decompose [7]. Nevertheless, this important advantage is obscured by the practical and formal difficulties that arise in the composition of transformations, where the corresponding two-forms are related by the Baker-Campbell-Hausdorff (BCH) formula. It is the absence of a simple and compact expression for the BCH formula that originates these difficulties. The main purpose of this paper is to obtain such an expression.

The simpler example of classical rotations illustrates clearly the above situation in the three-dimensional Euclidean case. Let us remember it. The nine components of a rotation matrix R may be related in more or less complicate forms to different relative parameterizations [8], but the intrinsic elements of the matrix are the rotation axis $\mathbf{u}$ and the rotation angle α. In the exponential domain, the element $\mathbf{r}$ of the rotation algebra corresponding to R decomposes in the form $\mathbf{r} = \alpha * \mathbf{u}$ with $\alpha = |\mathbf{r}|$. Here $*$ is the dual operator associated to the Euclidean metric δ so that, $\mathbf{u}$ being a vector, $*\mathbf{u}$ is a two-form. And $|\mathbf{r}| = (\mathbf{r}, \mathbf{r})^{\frac{1}{2}}$ is the *module* of $\mathbf{r}$, with $(\mathbf{r}, \mathbf{r}) = -\frac{1}{2}\, tr\, \mathbf{r}^2$ and $\mathbf{rs}$ is the induced product on two-forms (in local coordinates $(\mathbf{rs})_{\mu\nu} = \mathbf{r}_{\mu\tau}\mathbf{s}^{\tau}_{\nu}$). Thus, one has for R the expression:

$$R = \exp \mathbf{r} = \delta - \frac{\sin\alpha}{\alpha}\, \mathbf{r} + \frac{1-\cos\alpha}{\alpha^2}\, \mathbf{r}^2. \tag{1}$$

And conversely, starting from R, one obtains

$$\mathbf{r} = \log R = \frac{\arcsin \rho/2}{\rho} \left({}^t R - R\right) \tag{2}$$

where ρ is given by

$$\rho \equiv \sqrt{(1 + tr\, R)(3 - tr\, R)}$$

and ${}^tR$ denotes the transposed of R. In other words, the rotation angle α and the rotation axis $\mathbf{u}$ of a rotation matrix R are intrinsically given by

$$\cos\alpha = \frac{1}{2}(tr\,R - 1), \qquad \mathbf{u} = \frac{1}{\rho} * ({}^tR - R).$$

Suppose now that we have another rotation matrix S corresponding to the rotation vector $\beta\mathbf{v}$, that is to say, to the rotation algebra element $\mathbf{s} = \beta * \mathbf{v}$. The element $\mathbf{t}$ corresponding to the composed rotation $T = RS$ is given by the BCH-formula [9]:

$$\begin{aligned} \mathbf{t} &= \mathbf{r} \bullet \mathbf{s} \\ &= \frac{2\sigma}{\sin\sigma}\left\{\frac{1}{\alpha}\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\mathbf{r} + \frac{1}{\beta}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\mathbf{s} + \frac{1}{\alpha\beta}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}[\mathbf{r},\mathbf{s}]\right\}, \end{aligned} \tag{3}$$

where σ is given by

$$\cos\sigma = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \gamma\sin\frac{\alpha}{2}\sin\frac{\beta}{2},$$

γ being the cosine of the rotation axes, $\gamma = (\mathbf{u},\mathbf{v}) = \frac{1}{\alpha\beta}(\mathbf{r},\mathbf{s})$, and $[\mathbf{r},\mathbf{s}]$ being the Lie bracket of the two-forms $\mathbf{r}$ and $\mathbf{s}$, $[\mathbf{r},\mathbf{s}] = \mathbf{rs} - \mathbf{sr}$.

In other words, if $\mathbf{r}$ and $\mathbf{s}$ are two rotations corresponding to the rotation vectors $\alpha\mathbf{u}$ and $\beta\mathbf{v}$, the rotation angle θ and the rotation axis $\mathbf{w}$ of their composition $\mathbf{t}$, $\mathbf{t} = \mathbf{r} \bullet \mathbf{s} = \theta * \mathbf{w}$, are given by

$$\begin{aligned} \theta &= 2\sigma, \\ \mathbf{w} &= \frac{1}{\sin\sigma}\left\{\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\mathbf{u} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\mathbf{v} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\mathbf{u}\times\mathbf{v}\right\}, \end{aligned} \tag{4}$$

where $\mathbf{u}\times\mathbf{v}$ is the vector product.

In Minkowski space-time, the analogous of expression (1), that is to say, the general and covariant explicit form of local Lorentz transformations as exponential of two-forms, has been given in [10], although some partial results were already known [11]. Nevertheless, the analogous of expression (2) for the two-forms as logarithms of local Lorentz transformations seems not to have been considered but in [10].

Here we shall obtain the analogous of expressions (3) and (4) for Minkowski space-time, that is to say, the general and covariant expression of the Baker-Campbell-Hausdorff composition $\bullet$ of two two-forms as well as the relations between the intrinsic elements of this composition and those of its factor two-forms [12].

Our results are well adapted to theoretical considerations as well as to practical computations. They may be applied in all situations in which Lorentz transformations are implied, global ones in special relativity or local ones in both, special and general relativity. And this, not only for the above mentioned problems of adapted observers or synchronizations, but also in the study of special decompositions [13], Thomas precession [14], general equations of helices [15], motion of charged particles in particular electromagnetic fields [16], or the generalization of the binomial theorem [17].

These results may be also useful for heuristic researches in other fields. For example, in non linear electromagnetic theory. Physically, algebras are seen as first (tangent) approximations or weak (little) perturbations. This suggests a guiding idea for the search of non linear electromagnetic equations: to consider that the first object to be "nonlinearized" are not Maxwell equations for the electromagnetic field, but *the electromagnetic field* itself. Being today described by a two-form (element of the Lorentz algebra), the "finite" or "strong" description of the electromagnetic field would be given by a Lorentz field tensor, its exponential [18].

The paper is organized as follows. The computation of exponentials, logarithms and BCH compositions being easier in complex spaces, Section II is devoted to remember the real and complex elements that we shall need. Section III starts with the exponential of complex two-forms and obtains the general and covariant expression of the BCH composition of two complex two-forms as a linear combination of them and of their commutator, the coefficients being functions of the scalar invariants of the pair of two-forms. In Section IV the corresponding expression for the BCH composition of two real two forms is obtained, but now the duals of the two two-forms and of their commutator are generically necessary. Also, expressions for the eigenvalues and the invariant planes of the BCH composition are obtained in terms of the two two-forms.

II Preliminaries

We denote by $\Lambda^{\mathbb{C}}$ the complexification of Λ, space of two–forms on Minkowski space-time M_4: $\Lambda^{\mathbb{C}}$ is the complex linear space associated to $\Lambda \times \Lambda$ by the complex structure $J(F,G) = (-G,F)$ for $F, G \in \Lambda$. The element of $\Lambda^{\mathbb{C}}$ corresponding to the pair (F,G) is denoted by $\mathcal{A} = \frac{1}{2}(F + iG)$.

Thus, $\Lambda^{\mathbb{C}}$ is a $\mathbb{C}-$linear space of complex dimension 6 with two relevant

$\mathbb{C}-$linear subspaces

$$\begin{aligned}\Lambda^+ &= \{\mathcal{A} \in \Lambda^{\mathbb{C}} | *\mathcal{A} = i\mathcal{A}\} = \left\{\frac{1}{2}(F - i*F)|F \in \Lambda\right\}\\ \Lambda^- &= \{\mathcal{A} \in \Lambda^{\mathbb{C}} | *\mathcal{A} = -i\mathcal{A}\} = \left\{\frac{1}{2}(F + i*F)|F \in \Lambda\right\},\end{aligned}$$

where $*$ is the dual operator associated to the Lorentzian metric of M_4. It is verified that $\Lambda^{\mathbb{C}} = \Lambda^+ \oplus \Lambda^-$ and $\overline{\Lambda^+} = \Lambda^-$, so $\dim_{\mathbb{C}} \Lambda^+ = \dim_{\mathbb{C}} \Lambda^- = 3$.

As Λ has a structure of Lie algebra (with the commutator $[F,G] = FG - GF$, the product being defined in local coordinates by $(FG)_{\mu\nu} = F_{\mu\tau}G^{\tau}_{\nu}$), Λ^+ can be endowed with a $\mathbb{C}-$Lie algebra structure by the commutator in $\Lambda^{\mathbb{C}}$

$$[\mathcal{A},\mathcal{B}] = \frac{1}{4}([F,H] - [G,K]) + \frac{i}{4}([F,K] + [G,H]), \tag{5}$$

where $\mathcal{B} = \frac{1}{2}(H + iK)$. Then, $\Lambda^{\mathbb{C}}$ is a $\mathbb{C}-$Lie algebra and Λ^+ is a $\mathbb{C}-$Lie subalgebra of $\Lambda^{\mathbb{C}}$.

Similarly, Λ^+ is a $\mathbb{C}-$metric linear space with the $\mathbb{C}-$scalar product in $\Lambda^{\mathbb{C}}$

$$(\mathcal{A},\mathcal{B}) = \frac{1}{4}\{(F,H) - (G,K)\} + \frac{i}{4}\{(F,K) + (G,H)\}, \tag{6}$$

where $(\cdot,\cdot)$ in the second member is the induced scalar product in Λ, given by $(F,G) = -(1/2)\,tr(FG)$, tr being the trace operator.

It can be shown that every non vanishing $\mathcal{A} \in \Lambda^+$ admits a unique decomposition of the form $\mathcal{A} = \boldsymbol{u}\mathcal{U}$, where $\boldsymbol{u} \in \mathbb{C}$, $\mathrm{Re}(\boldsymbol{u}) \geq 0$ and $\mathcal{U} \in \Lambda^+$ with $(\mathcal{U},\mathcal{U}) \in \{-2, 0\}$, $\boldsymbol{u} = \frac{1}{2}$ when $(\mathcal{U},\mathcal{U}) = 0$. In this last case, $\mathcal{A}$ is called *null*, otherwise, it is called *regular*. In both cases, $\mathcal{U}$ is said the *geometry* of $\mathcal{A}$ (degenerate or regular 2×2 almost product structure, respectively), and the complex number $\boldsymbol{a} = \sqrt{-\frac{1}{2}(\mathcal{A},\mathcal{A})}$ is said its *invariant*; only when $\mathcal{A}$ is regular one has $\boldsymbol{a} = \boldsymbol{u}$.

If $\mathcal{A} \in \Lambda^+$, there is only one $F \in \Lambda$ such that $\mathcal{A} = \frac{1}{2}(F - i*F)$. When $\mathcal{A}$ is *regular*, so is called F and we have the decomposition $F = \alpha U - \widetilde{\alpha} * U$, with $\boldsymbol{a} = \frac{1}{2}(\alpha - i*\widetilde{\alpha})$ and $\mathcal{U} = U - i*U$. When $\mathcal{A}$ is *null*, so is also called F and we have $\boldsymbol{a} = 0$ and $\mathcal{U} = F - i*F$.

Let $\mathcal{A},\mathcal{B} \in \Lambda^+$. The complex number $\boldsymbol{k} = \frac{1}{2}(\mathcal{A},\mathcal{B})$ is said the *mixed invariant* of $\mathcal{A}$ and $\mathcal{B}$ and the set of invariants $\boldsymbol{a}$ (the invariant of $\mathcal{A}$), $\boldsymbol{b}$ (the invariant of $\mathcal{B}$) and $\boldsymbol{k}$ (the mixed invariant of $\mathcal{A}$ and $\mathcal{B}$) are called the *invariants of the pair* $\mathcal{A}$, $\mathcal{B}$. If $\mathcal{A} = \frac{1}{2}(F - i*F)$ and $\mathcal{B} = \frac{1}{2}(G - i*G)$, the mixed invariant $\boldsymbol{k}$ may be written $\boldsymbol{k} = \frac{1}{2}(\kappa - i\widetilde{\kappa})$, with $\kappa \equiv \frac{1}{2}(F,G)$

and $\widetilde{\kappa} \equiv \frac{1}{2}(F, *G)$. The following expression gives the relation between the invariant of $[\mathcal{A}, \mathcal{B}]$ and the invariants of the pair $\mathcal{A}$, $\mathcal{B}$

$$([\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{B}]) = 8(\boldsymbol{a}^2\boldsymbol{b}^2 + \boldsymbol{k}^2). \tag{7}$$

This result is obtained by a straightforward computation taking into account Lemma 3 of [19].

The space of second rank covariant tensors can be endowed with an associative algebra structure with identity element as well as a Lie algebra structure in the standard way; using linear extensions as in (5) and (6) the complexification of that space can also be endowed with an associative algebra structure with identity element and with a $\mathbb{C}-$Lie algebra structure. The expression for the product is

$$(M + iN)(P + iQ) = MP - NQ + i(MQ + NP).$$

¿From the identity $FG - *G * F = -(F, G)g$, with F and G in Λ, it is obtained for $\mathcal{A}$ and $\mathcal{B}$ in Λ^+

$$\mathcal{A}\mathcal{B} - *\mathcal{B} * \mathcal{A} = \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} = -(\mathcal{A}, \mathcal{B})g; \tag{8}$$

where the first equality is a consequence of the fact that $*\mathcal{A} = i\mathcal{A}$ for the elements of Λ^+.

III The Complex BCH–formula

In order to sum the exponential series of a complex two-form, let us consider the following lemma, which can be proven using (8) and induction over n.

Lemma 1 *For any two–form $\mathcal{A}$ of Λ^+ with invariant $\boldsymbol{a}$, one has*

$$\mathcal{A}^{2n} = \boldsymbol{a}^{2n} g \qquad \text{and} \qquad \mathcal{A}^{2n+1} = \boldsymbol{a}^{2n}\mathcal{A} \qquad (n \in \mathbb{N}).$$

Thus, when $\mathcal{A} \in \Lambda^+$, it is verified that

$$\exp \mathcal{A} = \sum_{n=0}^{\infty} \frac{\mathcal{A}^n}{n!} = \sum_{n=0}^{\infty} \frac{\mathcal{A}^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{\mathcal{A}^{2n+1}}{(2n+1)!} = \cosh \mathcal{A} + \sinh \mathcal{A};$$

last equality being a consequence of the definition of the hyperbolic sine and cosine of a matrix.

Introducing the entire complex functions ($\boldsymbol{z} \in \mathbb{C}$)

$$\mathrm{c}(\boldsymbol{z}) = \cosh \boldsymbol{z} \qquad \text{and} \qquad \mathrm{s}(\boldsymbol{z}) = \begin{cases} \dfrac{\sinh \boldsymbol{z}}{\boldsymbol{z}} & \boldsymbol{z} \neq 0 \\ 1 & \boldsymbol{z} = 0, \end{cases} \tag{9}$$

we have the following result.

Proposition 1 *For any two–form $\mathcal{A}$ of Λ^+ with invariant $\boldsymbol{a}$, one has*

$$\cosh\mathcal{A} = \mathrm{c}(\boldsymbol{a})g \qquad \textit{and} \qquad \sinh\mathcal{A} = \mathrm{s}(\boldsymbol{a})\mathcal{A}.$$

It follows from the definitions and lemma 1. Then, as a corollary we obtain next theorem.

Theorem 1 *For any two–form $\mathcal{A}$ of Λ^+ one has*

$$\exp\mathcal{A} = \mathrm{c}(\boldsymbol{a})g + \mathrm{s}(\boldsymbol{a})\mathcal{A},$$

where c and s are the functions (9) of the invariant $\boldsymbol{a}$ of $\mathcal{A}$ and g is the Lorentzian metric of the Minkowski space.

The real expression of the exponential for the Lorentz group [10] can be easily obtained from this result.

The element $\mathcal{C} \in \Lambda^+$ such that $\exp\mathcal{A}\exp\mathcal{B} = \exp\mathcal{C}$ or, equivalently, $\mathcal{C} = \log(\exp\mathcal{A}\exp\mathcal{B})$ is given by the well known BCH–formula; this defines the so called BCH composition

$$\mathcal{A}\bullet\mathcal{B} = \log(\exp\mathcal{A}\exp\mathcal{B}).$$

Denoting by $\boldsymbol{c}$ the invariant of $\mathcal{C} = \mathcal{A}\bullet\mathcal{B}$, one has

$$\begin{aligned}\exp\mathcal{C} &= \mathrm{c}(\boldsymbol{c})g + \mathrm{s}(\boldsymbol{c})\mathcal{C} = \exp\mathcal{A}\exp\mathcal{B} = (\mathrm{c}(\boldsymbol{a})g + \mathrm{s}(\boldsymbol{a})\mathcal{A})(\mathrm{c}(\boldsymbol{b})g + \mathrm{s}(\boldsymbol{b})\mathcal{B}) \\ &= \mathrm{c}(\boldsymbol{a})\,\mathrm{c}(\boldsymbol{b})g + \mathrm{s}(\boldsymbol{a})\,\mathrm{c}(\boldsymbol{b})\mathcal{A} + \mathrm{c}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b})\mathcal{B} + \mathrm{s}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b})\mathcal{A}\mathcal{B}.\end{aligned}$$

Its symmetric part gives

$$\cosh\mathcal{C} = \mathrm{c}(\boldsymbol{c})g = \mathrm{c}(\boldsymbol{a})\,\mathrm{c}(\boldsymbol{b})g + \frac{1}{2}\mathrm{s}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b})(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A})\,,$$

of which, taking its trace and remembering expression (8) and the definition of the mixed invariant $\boldsymbol{k}$, one obtains:

Proposition 2 *The invariant $\boldsymbol{c}$ of the BCH composition $\mathcal{C} = \mathcal{A}\bullet\mathcal{B}$ of two two–forms $\mathcal{A}$, $\mathcal{B}$ in $\in \Lambda^+$ is given by*

$$\cosh\boldsymbol{c} = \mathrm{c}(\boldsymbol{a})\,\mathrm{c}(\boldsymbol{b}) - \boldsymbol{k}\,\mathrm{s}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b}) \tag{10}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{k}$ are the invariants of the pair $\mathcal{A}$, $\mathcal{B}$.

Now, the antisymmetric part of $\exp\mathcal{C}$ gives the complex version of our main result.

Theorem 2 *The BCH composition $\mathcal{A}\bullet\mathcal{B}$ of two two–forms $\mathcal{A}$, $\mathcal{B}$ in $\in \Lambda^+$ is given by*

$$\mathcal{A}\bullet\mathcal{B} = \mathrm{s}(\boldsymbol{c})^{-1}\left\{\mathrm{s}(\boldsymbol{a})\,\mathrm{c}(\boldsymbol{b})\mathcal{A} + \mathrm{c}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b})\mathcal{B} + \frac{1}{2}\mathrm{s}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b})[\mathcal{A},\mathcal{B}]\right\}$$

where $\boldsymbol{c}$, is the invariant given by proposition 2.

IV The real BCH–formula for the Lorentz Group

To obtain the expression of the real BCH–formula for the Lorentz group we need the following proposition, whose proof is based on the fact that $[\mathcal{A},\mathcal{B}]=0$ whenever $\mathcal{A}\in\Lambda^+$ and $\mathcal{B}\in\Lambda^-$.

Proposition 3 *Let $F,G\in\Lambda$ and $\mathcal{A},\mathcal{B}\in\Lambda^+$ be such that $\mathcal{A}=\frac{1}{2}(F-i*F)$ and $\mathcal{B}=\frac{1}{2}(G-i*G)$, then*

$$F\bullet G = 2\,\mathrm{Re}(\mathcal{A}\bullet\mathcal{B}).$$

Proof: As, by definition

$$\exp F\bullet G=\exp F\exp G$$

then

$$\begin{aligned}\exp F\bullet G &= \exp(\mathcal{A}+\overline{\mathcal{A}})\exp(\mathcal{B}+\overline{\mathcal{B}})=\exp\mathcal{A}\exp\overline{\mathcal{A}}\exp\mathcal{B}\exp\overline{\mathcal{B}}\\ &= \exp\mathcal{A}\exp\mathcal{B}\exp\overline{\mathcal{A}}\exp\overline{\mathcal{B}}=\exp\mathcal{A}\exp\mathcal{B}\,\overline{\exp\mathcal{A}\exp\mathcal{B}}\\ &= \exp\mathcal{A}\bullet\mathcal{B}\,\overline{\exp\mathcal{A}\bullet\mathcal{B}}=\exp\mathcal{A}\bullet\mathcal{B}\exp\overline{\mathcal{A}\bullet\mathcal{B}}\\ &= \exp\left((\mathcal{A}\bullet\mathcal{B})\bullet\overline{(\mathcal{A}\bullet\mathcal{B})}\right)=\exp\left((\mathcal{A}\bullet\mathcal{B})+\overline{(\mathcal{A}\bullet\mathcal{B})}\right);\end{aligned}$$

second equality is due to the fact that they commute, last equality is because the only terms of the BCH–series of any elements $\mathcal{A}$ and $\mathcal{B}$ with $[\mathcal{A},\mathcal{B}]=0$ is $\mathcal{A}+\mathcal{B}$. □

Let us define the complex functions

$$\begin{aligned}\boldsymbol{f} &= \mathrm{s}(\boldsymbol{c})^{-1}\,\mathrm{s}(\boldsymbol{a})\,\mathrm{c}(\boldsymbol{b})\\ \boldsymbol{g} &= \mathrm{s}(\boldsymbol{c})^{-1}\,\mathrm{c}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b})\\ \boldsymbol{h} &= \frac{1}{2}\,\mathrm{s}(\boldsymbol{c})^{-1}\,\mathrm{s}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b}),\end{aligned}\qquad(11)$$

of the invariants of the pair $\mathcal{A}$, $\mathcal{B}$ and denote for short x and $\widetilde{x}$ the real and imaginary parts respectively of any complex function $\boldsymbol{x}$. A straightforward computation gives formally the main result for the real case as a corollary of the previous theorem.

Theorem 3 *The BCH composition $F\bullet G$ of two real two–forms F and G on Minkowski space, is given by*

$$F\bullet G=f\,F+g\,G+h\,[F,G]+\widetilde{f}\,*F+\widetilde{g}\,*G+\widetilde{h}\,*[F,G],$$

where the functions f, g, h, and $\widetilde{f}$, $\widetilde{g}$, $\widetilde{h}$, are respectively the real and imaginary parts of the functions (11) of the invariants of the pair $(1/2)(F-i*F)$, $(1/2)(G-i*G)$.

These coefficients are functions of the real invariants $(\alpha, \widetilde{\alpha})$, $(\beta, \widetilde{\beta})$ and $(\kappa, \widetilde{\kappa})$ of the pair of two–forms F, G. Our goal now is to find direct expressions of them.

For this purpose, let us first associate, to the sole invariants of F and G, the three following groups of quantities: their *normalized* values

$$
\begin{aligned}
&\underline{\alpha} = \frac{\alpha}{\alpha^2 + \widetilde{\alpha}^2} \quad , \quad \underline{\widetilde{\alpha}} = \frac{-\widetilde{\alpha}}{\alpha^2 + \widetilde{\alpha}^2} \quad , \\
&\underline{\beta} = \frac{\beta}{\beta^2 + \widetilde{\beta}^2} \quad , \quad \underline{\widetilde{\beta}} = \frac{-\widetilde{\beta}}{\beta^2 + \widetilde{\beta}^2} \quad ,
\end{aligned}
\tag{12}
$$

their *crossed* values

$$
\delta = \underline{\alpha}\underline{\beta} - \underline{\widetilde{\alpha}}\underline{\widetilde{\beta}} \quad , \quad \widetilde{\delta} = \underline{\alpha}\underline{\widetilde{\beta}} + \underline{\widetilde{\alpha}}\underline{\beta} \quad , \tag{13}
$$

and their trigonometric functions

$$
\begin{aligned}
Cc^{\pm} &= \cosh \tfrac{\alpha+\beta}{2} \cos \tfrac{\widetilde{\alpha}+\widetilde{\beta}}{2} \pm \cosh \tfrac{\alpha-\beta}{2} \cos \tfrac{\widetilde{\alpha}-\widetilde{\beta}}{2} \,, \\
Ss^{\pm} &= \sinh \tfrac{\alpha+\beta}{2} \sin \tfrac{\widetilde{\alpha}+\widetilde{\beta}}{2} \pm \sinh \tfrac{\alpha-\beta}{2} \sin \tfrac{\widetilde{\alpha}-\widetilde{\beta}}{2} \,, \\
Cs^{\pm} &= \cosh \tfrac{\alpha+\beta}{2} \sin \tfrac{\widetilde{\alpha}+\widetilde{\beta}}{2} \pm \cosh \tfrac{\alpha-\beta}{2} \sin \tfrac{\widetilde{\alpha}-\widetilde{\beta}}{2} \,, \\
Sc^{\pm} &= \sinh \tfrac{\alpha+\beta}{2} \cos \tfrac{\widetilde{\alpha}+\widetilde{\beta}}{2} \pm \sinh \tfrac{\alpha-\beta}{2} \cos \tfrac{\widetilde{\alpha}-\widetilde{\beta}}{2} \,.
\end{aligned}
\tag{14}
$$

With them, and taking into account the relation between hyperbolic and the trigonometric functions, one obtains

$$
\begin{aligned}
2\,\mathrm{c}(\boldsymbol{a})\,\mathrm{c}(\boldsymbol{b}) &= Cc^{+} - i\,Ss^{+} \,, \\
\mathrm{s}(\boldsymbol{a})\,\mathrm{c}(\boldsymbol{b}) &= (\underline{\alpha} Sc^{+} - \underline{\widetilde{\alpha}} Cs^{+}) - i(\underline{\alpha} Cs^{+} + \underline{\widetilde{\alpha}} Sc^{+}) \,, \\
\mathrm{c}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b}) &= (\underline{\beta} Sc^{-} - \underline{\widetilde{\beta}} Cs^{-}) - i(\underline{\beta} Cs^{-} + \underline{\widetilde{\beta}} Sc^{-}) \,, \\
\tfrac{1}{2}\,\mathrm{s}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b}) &= (\delta\, Cc^{-} - \widetilde{\delta}\, Ss^{-}) - i(\delta\, Ss^{-} + \widetilde{\delta}\, Cc^{-}) \,.
\end{aligned}
\tag{15}
$$

On the other hand, according to proposition 2, the complex invariant $\boldsymbol{c} = \frac{1}{2}(\gamma - i\widetilde{\gamma})$ is given by (10), which, taking into acount (15) and writing $\cosh \boldsymbol{c} \equiv \frac{1}{2}(e - i\widetilde{e})$, gives

$$
e = Cc^{+} + \chi Cc^{-} + \widetilde{\chi} Ss^{-} \quad , \quad \widetilde{e} = Ss^{+} - \widetilde{\chi} Ss^{-} + \chi Cc^{-} \,, \tag{16}
$$

where the χ's are defined by

$$
\chi = 4(\kappa\,\delta + \widetilde{\kappa}\,\widetilde{\delta}) \; \textit{quad}, \qquad \widetilde{\chi} = 4(\kappa\,\widetilde{\delta} - \widetilde{\kappa}\,\delta) \,.
$$

In order to determine the invariants $(\gamma, \widetilde{\gamma})$ of $F \bullet G$ as well as the real ingredients of $1/\,\mathrm{s}(\boldsymbol{c})$, let us introduce the three functions of the e's

$$\Gamma = \frac{1}{4}(e^2 + \widetilde{e}^2) \ , \ \ \Delta = \sqrt{(\Gamma+1)^2 - e^2} \ , \ \ \Theta = \frac{1}{4}(e^2 - \widetilde{e}^2) - 1 \ . \tag{17}$$

We shall see in the following that the *relative position* of the geometries of F and G, measured by the mixed invariants κ and $\widetilde{\kappa}$, appears in the BCH product $F \bullet G$ only across the pair of weighted scalars χ and $\widetilde{\chi}$; these weighted scalars appear in turn only across the above e's, which at the same time appear only by means of the three above scalars Γ, Δ and Θ.

Denoting by ϵ and $\widetilde{\epsilon}$ respectively the signs of the above scalars e and $\widetilde{e}$ one finds

$$\cosh\frac{\gamma}{2} = (1/\sqrt{2})\sqrt{\Gamma+\Delta+1}, \quad \cos\frac{\widetilde{\gamma}}{2} = \epsilon(1/\sqrt{2})\sqrt{\Gamma-\Delta+1},$$
$$\sinh\frac{\gamma}{2} = (1/\sqrt{2})\sqrt{\Gamma+\Delta-1}, \quad \sin\frac{\widetilde{\gamma}}{2} = \widetilde{\epsilon}(1/\sqrt{2})\sqrt{-\Gamma-\Delta+1},$$

so that we have:

Theorem 4 *The invariants γ and $\widetilde{\gamma}$ of the BCH composition two–form $F \bullet G$ are given by*

$$\cosh\gamma = \Gamma + \Delta, \qquad \cos\widetilde{\gamma} = \Gamma - \Delta,$$

where $\mathrm{sign}(\sin\widetilde{\gamma}) = -\epsilon\,\widetilde{\epsilon}$ *and* Γ *and* Δ *are the functions* (17) *of the invariants of the pair F, G.*

This theorem and relations (16) give the following result:

Corollary 1 *The invariants of the BCH compositions $F \bullet G$ and $G \bullet F$ of two two–forms F and G coincide.*

The above theorem allows to know $\boldsymbol{c} = \frac{1}{2}(\gamma - i\widetilde{\gamma})$ and $\cosh \boldsymbol{c} = \frac{1}{2}(e - i\widetilde{e})$ and consequently $\mathrm{s}(\boldsymbol{c})$. Denoting respectively by Σ and $\widetilde{\Sigma}$ the real and imaginary parts of $\mathrm{s}(\boldsymbol{c})^{-1}$, one obtains

Corollary 2 *In terms of the invariants of the pair F, G, the functions Σ and $\widetilde{\Sigma}$ are given by*

$$\begin{aligned} \Sigma &= \frac{1}{2\sqrt{2}\Delta}(\epsilon\,\gamma\,\sqrt{\Delta+\Theta} + \widetilde{\epsilon}\,\widetilde{\gamma}\,\sqrt{\Delta-\Theta}), \\ \widetilde{\Sigma} &= \frac{1}{2\sqrt{2}\Delta}(\widetilde{\epsilon}\,\gamma\,\sqrt{\Delta-\Theta} - \epsilon\,\widetilde{\gamma}\,\sqrt{\Delta+\Theta}), \end{aligned} \tag{18}$$

where $\gamma = \arg\cosh{(\Gamma+\Delta)}$, $\widetilde{\gamma} = \arg\cos{(\Gamma-\Delta)}$.

Thus, from (18), (15), (11) and theorem 3, one has the main result:

Theorem 5 *The BCH composition $F \bullet G$ of two two–forms F and G is of the form*

$$F \bullet G = f\,F + g\,G + h\,[F,G] + \widetilde{f} * F + \widetilde{g} * G + \widetilde{h} * [F,G]$$

where

$$\begin{aligned}
f &= (\underline{\alpha}\Sigma + \widetilde{\underline{\alpha}}\widetilde{\Sigma})Sc^{+} + (\underline{\alpha}\widetilde{\Sigma} - \widetilde{\underline{\alpha}}\Sigma)Cs^{+}\ , \\
g &= (\underline{\beta}\Sigma + \widetilde{\underline{\beta}}\widetilde{\Sigma})Sc^{-} + (\underline{\beta}\widetilde{\Sigma} - \widetilde{\underline{\beta}}\Sigma)Cs^{-}\ , \\
h &= (\delta\Sigma + \widetilde{\delta}\widetilde{\Sigma})Cc^{-} + (\delta\widetilde{\Sigma} - \widetilde{\delta}\Sigma)Ss^{-}\ , \\
\widetilde{f} &= (\underline{\alpha}\widetilde{\Sigma} - \widetilde{\underline{\alpha}}\Sigma)Sc^{+} - (\underline{\alpha}\Sigma + \widetilde{\underline{\alpha}}\widetilde{\Sigma})Cs^{+}\ , \\
\widetilde{g} &= (\underline{\beta}\widetilde{\Sigma} - \widetilde{\underline{\beta}}\Sigma)Sc^{-} - (\underline{\beta}\Sigma + \widetilde{\underline{\beta}}\widetilde{\Sigma})Cs^{-}\ , \\
\widetilde{h} &= (\delta\widetilde{\Sigma} - \widetilde{\delta}\Sigma)Cc^{-} - (\delta\Sigma + \widetilde{\delta}\widetilde{\Sigma})Ss^{-}\ ,
\end{aligned} \tag{19}$$

the normalized invariants, the quantities δ and $\widetilde{\delta}$, Σ and $\widetilde{\Sigma}$, and the variables $x^{\pm}$ being respectively given by (12)*,* (13)*,* (18) *and* (14)*.*

One can directly obtain the geometry of $F \bullet G$ as a function of the pair F, G. In the null case, the two–form and its geometry are proportional, so that we have only to study the regular case. Both cases are discriminated by the value of $\mathrm{c}(\boldsymbol{a})\,\mathrm{c}(\boldsymbol{b}) - \boldsymbol{k}\,\mathrm{s}(\boldsymbol{a})\,\mathrm{s}(\boldsymbol{b})$; from Theorem 2, this value is 1 only when $F \bullet G$ is null or 0. Taking into account Proposition 3 and the definition of the geometry $\{W, *W\}$, $F \bullet G = \gamma W - \widetilde{\gamma} * W$, one has

$$\mathcal{A} \bullet \mathcal{B} = \frac{1}{2}(\gamma - i\widetilde{\gamma})(W - i * W);$$

then, from theorem 2 one obtains

Theorem 6 *The element W of the geometry $\{W, *W\}$ of the BCH composition $F \bullet G$ of two two-forms F and G, is given by the same expression of Theorem 5 for $F \bullet G$ but where the functions Σ and $\widetilde{\Sigma}$ are substituted respectively by*

$$\frac{1}{2}\left(\Sigma - \widetilde{\Sigma}\right) \quad \textit{and} \quad \frac{1}{2}\left(\Sigma + \widetilde{\Sigma}\right).$$

All the formulas given here are general, i.e. valid for *any* pair of two–forms. This fact “justify” their relatively complicate expressions. Fortunately, in contrast with the three–dimensional rotation group, that only admits a one-dimensional subgroup (rotations about same axis), Lorentz group admits thirteen proper subgroups, and we have shown elsewhere [20] that *twelve* of them may be generated by a pair of two–forms. In all these twelve cases, the above BCH formulas simplify notably. The corresponding expressions will be given elsewhere [21].

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References

[1] Here a *local Lorentz transformation* means a field of Lorentz transformations of the tangent space at *each* point.

[2] That is to say, to observers whose velocity vector is tangent to the curves of the congruence.

[3] A *synchronization* is a specification of the locus of points (hypersurfaces) of equal time. Every synchronization has a *natural* set of observers: those whose velocity vector is normal to the family of hypersurfaces.

[4] They are given biunivocally only for *exponential* groups (J. Dixmier, Bull. Soc. math. France, **85**, 113 (1957); L. Pukanszky, Trans. Amer. Math. Soc., **126**, 487 (1967).

[5] Because of the space-time metric, the elements of the algebra may be written as second order antisymmetric covariant tensors at each point so that, in the corresponding domain of the space-time, they define a two-form.

[6] The usual representation of Lorentz transformations by matrices or second order tensors carries an excessive number of nonstrict quantities, namely $n^2 = 16$ components, which hides their *intrinsic elements*. These elements depend only on $n(n-1)/2 = 6$ parameters, the group dimension, and are those in which the Lorentz transformations may be *biunivocally* and *covariantly* decomposed. The intrinsic elements of a Lorentz transformation are thus its non space-like invariant 2-plane, and its two eigenvalues. As it is well known, in the regular case one of these eigenvalues is a hyperbolic angle, and gives the magnitude of the *proper* boost on the timelike invariant 2-plane. The other eigenvalue is a trigonometric angle, and fixes the rotation on the orthogonal space-like invariant 2-plane. Intrinsic elements have not to be confused with velocity-rotation relative parametrizations, for which biunivocity fails.

[7] See next Section.

[8] Euler angles, Cayley-Klein parameters, etc.

[9] For the original articles see: J.E. Campbell, Proc. London Math. Soc., **28**, 381 (1897); **29**, 14 (1898); H. F. Baker, *ibid*, **34**, 347 (1902); **2**, 293 (1904); **3**, 24 (1905); F. Hausdorff, Ber. Verhandl. Saechs. Akad. Wiss. Leipzig, Math. Naturw. Kl., **58**, 19 (1906).

[10] B. Coll and F. San José, Gen. Relativ. Gravit., **22**, 811 (1990).

[11] A. H. Taub, Phys. Rev., **73**, 786 (1948); S. L. Bazanski, J. Math. Phys., **6**, 1201 (1965);

[12] It is to be noted that the product of two Lorentz transformations in terms of *relative* parameters (namely, relative velocity and relative rotation), which is well known from long time (see, for exemple M. Rivas *et al.*, Eur. J. Phys., **7**, 1 (1986)), differs strongly from the BCH product. This is due to the facts that in the case of relative parametrizations every factor is framed in a different basis (i.e. for different observers) and that the corresponding parametrizations refers to these different bases. A connection between the relative product and the BCH formula not only needs the relation between relative parameters and intrinsic elements, but *also* the relation between the relative parameters with respect to different observers, involving notions such as "velocity of a point with respect to an observer *as seen* by another observer". We shall not consider here such a connection.

[13] See for example C. B. van Wyk, J. Math. Phys., **32**, 425 (1991).

[14] Corrections to the Tomas precesion (L. H. Thomas, *Nature*, **117**, 514 (1926)) have been obtained from the BCH formula by N. Salingaros, J. Math. Phys., **25**, 706 (1984).

[15] The equations of relativistic helices are due to J. L. Synge, Proc. Roy. Irish Acad., sec. A, **65**, 27 (1967); their expressions may by manifestly simplified using the results presented here.

[16] The first to consider this problem in general was A. H. Taub, Phys. Rev., **73**, 786 (1948); and an intrinsic characterization in term of Frenet-Serret parameters was given by E. Honig *et al.*, J. Math. Phys., **15**, 774 (1974).

[17] See J. Morales and A. Flores-Riveros, J. Math. Phys., **30**, 393 (1989).

[18] The analysis of this idea needs, beside our results, their reciprocals: the internal operations on the group of Lorentz tensors that correspond by the exponential to the two internal operations of its algebra (addition and commutator). But this is another affair.

[19] B. Coll, F. San José Martínez; J. Math. Phy., vol. **36**, 4350 (1995).

[20] B. Coll and F. San José, *On the algebras generated by two 2-forms in Minkowski space-time*, J. Math. Phys., **37**, 5792 (1996); see also *Relative position of a pair of planes and algebras generated by two 2-forms in relativity*, in *Recent Developments in Gravitation*, World Scientific, 1991, p. 210.

[21] B. Coll, F. San José Martínez; *Particular pairs of Lorentz Transformations*, in preparation.